

**Columbia University  
in the City of New York**

FINAL REPORT

Contract NAS8-32212

Principal Investigators:

Peter W. Likins  
Richard W. Longman

School of Engineering and Applied Science  
Columbia University  
New York, New York 10027



November, 1979

## TABLE OF CONTENTS

	Page
OVERVIEW	iv
NOMENCLATURE	viii
LIST OF TABLES	ix
LIST OF FIGURES	x
PART I: THE DEGREE OF CONTROLLABILITY CONCEPT AND ITS APPLICATION IN THE LOCATION OF ACTUATORS ON VERY LARGE FLEXIBLE SPACECRAFT	xi
1. INTRODUCTION	1
2. THEORY	5
2.1 Introduction	5
2.2 Definition of the Degree of Controllability	8
2.3 Concepts for Approximating the Recovery Region	16
2.4 Fundamental Equation for Recovery Region Approximation in the Multiple Control Case	23
2.4.1 Reduction of $\delta$ -Equation to the Spatial Form	26
2.4.2 Reduction of $\delta$ -Equation to the Real Domain	28
2.4.3 Summary	35
2.5 Test for Complete Controllability of the System	38
2.5.1 Characteristics of the Spatial Form from the Point of View of Controllability	38
2.5.2 Nonsingularity of the C Matrix	41

2.5.3	Necessary and Sufficient Conditions for Complete Controllability of Linear Time Invariant Systems	45
2.6	Approximation for Recovery Region	65
2.7	Summary	77
3.	ANALYSIS OF LARGE FLEXIBLE SPACECRAFT	78
3.1	Introduction	78
3.2	A Typical Planar Motion Model of a Very Large Flexible Spacecraft	79
3.3	Dynamics	80
3.3.1	Kinetic Energy	83
3.3.2	Potential Energy	86
3.3.3	Generalized Forces	87
3.3.4	Equations of Motion	90
3.3.5	Linearized Equations of Motions	91
3.3.6	Normalized Control Effort	93
3.3.7	Normalized State Space Form for System Equations	95
3.4	Summary	104
4.	APPLICATIONS TO A SPECIFIC MODEL OF A LARGE FLEXIBLE SPACECRAFT	105
4.1	Model Description	106
4.2	Degree of Controllability	107
4.2.1	Mode Shape Integrals	107
4.2.2	Computation of Inertia and Stiffness Matrices	111

4.2.3	Computation of Matrices for Generalized Forces	114
4.2.4	State Space Form for System Equations	118
4.3	Approximate Degree of Controllability	125
5.	RESULTS AND DISCUSSIONS	136
5.1	Introduction	136
5.2	Influence Curves for Actuator Locations	137
5.3	Single Force Actuator	143
5.3.1.	Introduction	143
5.3.2	Symmetric Mode of $B_F$	144
5.3.3	Antisymmetric Mode of $B_F$	146
5.4	Single Torque Actuator	150
5.4.1	Introduction	150
5.4.2	Symmetric Mode of $B_F$	150
5.4.3	Antisymmetric Mode of $B_F$	153
5.5	Summary for a Single Actuator	157
5.6	Multiple Actuators	158
5.7	Summary	163
6.	SUMMARY AND CONCLUSIONS	165
7.	REFERENCES	182
8.	APPENDICES	183
	Appendix A: Matrix Algebra	183
	Appendix B: A Definition of the Degree of Controllability -- A Criterion for Actuator Placement	186
PART II:	INSTRUMENT FLEXIBILITY LIMITATIONS FOR STABILITY OF MULTI-PURPOSE SPACECRAFT WITH FIXED CONTROL SYSTEM DESIGN	203

## OVERVIEW

In October of 1976 Columbia University undertook an investigation of a class of space vehicle control problems relating to vehicle or component flexibility. The sole Principal Investigator initially was Peter Likins, Professor and Dean of the School of Engineering and Applied Science. In August of 1978 Professor Richard Longman joined the project as Co-Principal Investigator. Under the direction of the two co-principal investigators, the substantial tasks of this study were largely the responsibility of Mr. Chittur Viswanathan, and the report that follows is the principal content of his Ph.D. dissertation. Appendix B of Part I of this report (see Section 8) is a technical paper written by Professor Longman for journal publication, and as such it may be read independently of the body of the report.

The study progressed in two phases, represented by Parts I and II of the Final Report. Part II is submitted here in more abbreviated form than is planned for the Viswanathan dissertation, both because of the size of the total documentation package and because of delays in preparation of the report. The expanded presentation of Part II from the dissertation includes an appendix of derivations, available upon request.

In technical content the study addresses two quite distinct engineering questions:

Part I. Where should actuators be placed on spacecraft with distributed flexibility?

Part II. How should we specify the dynamic characteristics of flexible instruments to be pointed by a prescribed control system?

The class of problem considered in Part I is of wide-ranging concern, since it is raised by many proposed spacecraft; the Solar Power Satellite is perhaps the most dramatic example.

The class of problem treated in Part II is of specific applicability to the Instrument Pointing System on the Space Shuttle. In this application an automatically controlled gimbal mechanism is designed to point a variety of interchangeable instruments toward their targets, with the gimbal system mounted on an actively controlled spacecraft. Future instruments must be designed after the control system characteristics are established.

Parts I and II of this report describe work performed in reverse chronological order. The task initially undertaken (Part II) was an attempt to apply the methods of parameter plane stability analysis to the Shuttle-based Instrument Pointing Mount (IPM). By establishing stability boundaries in the plane defined by two design parameters of the pointed instrument, one might hope to provide the instrument designer with a useful tool for the selection of these two design parameters within the range for which the control system is stable. This concept (originally suggested by Dr. Sherman Seltzer, then of the NASA Marshall Space Flight Center) is easily implemented for sufficiently simple mathematical models of the spacecraft system, and the resulting parameter plane plots (see Part II) offer the prospect of genuine utility for preliminary design. However,

attempts to extend the useful range of the method to realistically complex mathematical models are less successful. The parameter plane approach can still be applied to system equations of considerable complexity, but the parameters indicated for the portrayal of stability boundaries are less useful for practical design.

After exploring the parameter plane concept for the IPM to what seem to be its practical limits, the Columbia group turned its energies to the actuator placement problem. The results of this effort (Part I following) are very encouraging, although not yet definitive.

In the design of the spacecraft of the past and present, the question of actuator (and sensor) location has received remarkably little attention. It has generally been assumed that both sensors and actuators should be placed on the essentially rigid, central body that comprises the core of most spacecraft of this era. In those exceptional cases in which sensors and actuators have instead been attached to flexible appendages, the objective has been to address specific physically defined requirements. For example, gas jets were located at the tips of the solar panels for the Mariner spacecraft series in order to maximize the "lever arm" producing a moment about the mass center of the spacecraft; the flexibility of the solar panels was ignored in this selection.

Future spacecraft may differ significantly from their antecedents in both scale and the distribution of flexible components; in some cases it may be necessary to treat the entire spacecraft as a flexible body, with no rigid, central core. In such cases, it becomes quite unclear where one should locate sensors or actuators, or even how many should be employed. If one compares two alternative designs, it is not even clear what criteria one should employ in comparisons.

The chief contribution of Part I is the development of a criterion for the comparative evaluation of alternative actuator locations. This criterion is a measure of a quantity introduced here as the "degree of controllability."

In what follows a definition is generated for the "degree of controllability" concept (after alternative definitions are considered and discarded); techniques are established for calculating this measure; and application is made to flexible spacecraft, both generically and specifically.

Although it would be unrealistic to argue that the criterion advanced here provides a unique measure of the quality of an actuator distribution design choice, it does seem to be as good a basis for evaluation as any measure yet devised.



# NOMENCLATURE

$A$	System matrix
$B$	Control matrix; body frame of axes
$C$	Spatial matrix
$E$	Unit matrix
$J$	Jordan canonical form
$K, \tilde{K}$	Stiffness matrices
$L$	Appendage Length
$M, \tilde{M}$	Inertia matrices
$N$	Number of appendage modes
$N_i, \bar{N}_i$	Weighting factors for modes
$n$	Dimension of state vector
$P$	Transformation matrix
$u$	control vector
$x$	State vector
$\Phi$	State transition matrix; matrix of mode shape functions
$\delta$	State space displacement
$\omega_i$	Appendage frequencies
$\theta$	Attitude
$\eta$	Appendage modal coordinates
$\xi$	Vehicle normal modes
$C, C^*$	Degree of controllability; approximate degree of controllability
$R, R^*$	Recovery region; approximate recovery region

## LIST OF TABLES

	221
1. Appendage Modal Constants	170
2. Data for Specific Model of a Large Flexible Spacecraft	171
3. Input System Data	172
4. System Constants	172
5. Locations of zeroes of Influence Curves for Force Actuators	173
6. Locations of zeroes of Influence Curves for Torque Actuators	173

ORIGINAL PAGE IS  
OF POOR QUALITY

# LIST OF FIGURES

	page
1. The Recovery Region and Its Rectangular Approximation	174
2. Parallelogram Bound on the Recovery Region	174
3. Improving the Approximation to the Recovery Region	175
4. A Typical Planar Motion Model of a Large Flexible Spacecraft	176
5. A Specific Model of a Very Large Flexible Spacecraft	176
6. Spacecraft in Symmetric Mode of Motion of Flexible Appendages	177
7. Spacecraft in Antisymmetric Mode of Motion of Flexible Appendages	177
8. Influence Curves for Force Actuators for Spacecraft in Symmetric Motion	178
9. Influence Curves for Torque Actuators for Spacecraft in Symmetric Motion	179
10. Influence Curves for Force Actuators for Spacecraft in Antisymmetric Motion	180
11. Influence Curves for Torque Actuators for Spacecraft in Antisymmetric Motion	181

## PART I

### THE DEGREE OF CONTROLLABILITY CONCEPT AND ITS APPLICATION IN THE LOCATION OF ACTUATORS ON VERY LARGE FLEXIBLE SPACECRAFT

## 1. INTRODUCTION

This research addresses problems broadly connected with the dynamics and control of flexible spacecraft. From the dynamics point of view, the modeling of a spacecraft consists of treating some components as rigid bodies and some as flexible. Thus, solar panels, booms, etc. must usually be modeled as flexible, but they may have a rigid core as a base. The sizes of the spacecraft can vary tremendously according to their intended purpose. For instance, one design for a solar power station (SPS) satellite calls for a 4.9 km x 14 km solar array. In such applications, it will be necessary to control not only the attitude, i.e., orientation, but also the shape of the various components of the spacecraft. The need for shape control arises due to the flexibility of the spacecraft or its components. Stringent requirements may be imposed on the accuracy of attitude and shape control. For instance, accuracy of the order of thousandths of an arcsecond are sometimes needed in the attitude control of certain scientific instruments. On some future spacecraft which function as large antennas, accurate shape control will be necessary to obtain distortion free signals. The accuracy requirement is a function of the wavelength of the signal transmitted from the antenna. In order to achieve this goal for large spacecraft with distributed flexibility, sensors and actuators must be distributed at various locations in the spacecraft.

ORIGINAL PAGE IS  
OF POOR QUALITY

From the control point of view, spacecraft can often be classified into two main groups:

1. Those for which the spacecraft is fully defined before the control system is designed; and
2. Those for which the control system must be specified before certain interchangeable parts of a multi-purpose spacecraft are selected for future missions.

In what follows consideration is given to both classes of problems.

Spacecraft in the first class includes the usual case, and the design of stable control systems for conventional spacecraft is a mature discipline. However, the extreme size and distributed flexibility of many future spacecraft pose various challenging new problems in their attitude and shape control. One such problem is the development of a rationale for the distribution of sensors and actuators throughout the entire body of the spacecraft. How to compare different actuator distributions and what criterion to apply to be able to decide on a particular choice of distribution of actuators are the specific objectives in Part I of this work.

A typical example in the second class is the Space Shuttle's Instrument Pointing System (IPS) which must be designed to accurately control the pointing of any of a family of scientific instruments, some of which have yet to be designed. These instruments are typically lightweight and their flexibility effects are important due to pointing requirements. The actual instruments involved and their purpose will vary from one shuttle flight to another. A spacecraft compatible with any one of such a set of instruments becomes a multi-purpose spacecraft.

But it is not possible or practicable to design a control system for every instrument the spacecraft might carry, since some of these instruments may have to be planned for future needs. What is done, instead, is to predesign a comprehensive control system for attitude control of a rigid spacecraft. This control system is specified for the corresponding multi-purpose spacecraft. And the objective is to seek the characteristics in the most general terms of those flexible instruments the given control system can control with appropriate stability margins.

This research work is organized in two parts. Part I deals with spacecraft belonging to the first group, in which the system is fully known and the distribution of actuators in the design of control systems is the subject. Part II deals with spacecraft belonging to the second group, in which the system is not fully known and the characteristics of the class of flexible instruments compatible with a given control system are investigated from the stability point of view.

PART I

THE DEGREE OF CONTROLLABILITY CONCEPT AND ITS  
APPLICATION IN THE LOCATION OF ACTUATORS  
ON VERY LARGE FLEXIBLE SPACECRAFT



## 2. THEORY

ORIGINAL PAGE IS  
OF POOR QUALITY

### 2.1 Introduction

In Part I attention is focused on an aspect of design of the control systems whose purpose is to control effectively the attitude and shape of very large flexible spacecraft. In the past, the approach to the control system design was usually based on the philosophy that a spacecraft was essentially a rigid body with the flexibility of antennas, booms, solar panels, etc. treated as extraneous disturbances. The purpose of the control was only attitude control of the spacecraft and the design was based on the rigid spacecraft with adjustments made for the influence of the attached flexible components on the behavior of the entire spacecraft. The term Dynamic Interaction was then used to describe this influence due to the component vibrations and localized energy dissipation, which were treated as second order phenomena even though occasionally they proved to be destabilizing for the entire spacecraft. A good historical review regarding the prevailing attitudes in this field for the last couple of decades can be found in [1]. Today, the planners in the spacecraft industry are contemplating structures of huge dimensions (of the order of kilometers) for future spacecraft of the 1990's or 2000 and beyond. For these vehicles, flexibility can no longer be considered a second order phenomenon and must be treated as an intrinsic property of the entire spacecraft. In

6

designing control systems for these spacecraft the objective is no longer just the attitude control but simultaneously the shape control of these huge structures. By shape control we mean the restoration of these huge structures to their nominal shape whenever they are disturbed due to any cause (thermal stresses, gravity gradient torque, aerodynamic torque, solar wind, etc. etc.). This is done by controlling the vibrations of a finite number of modes of a truly infinite set of modes which describe the physical shapes of these distributed parameter systems. Because of the strong coupling of the attitude dynamics and vibration modes, their control must be treated simultaneously. Classical control theory is unsuitable for such large dimensional multiple input multiple output systems, and it becomes essential to adopt modern multivariable control theory techniques.

In order to achieve attitude and shape control it will be necessary to distribute actuators throughout the entire body of the spacecraft. How should the number and locations of actuators be chosen in order to best control the flexible spacecraft? This problem has been recognized for some time, but to date little has appeared in the literature that would help guide the control system designer in placing the actuators. Most of the known results identify the minimum number of actuators needed for a given set of modes to be controlled, and identify certain specific actuator locations which cannot be used because they result in an uncontrollable system.

Once the designer chooses a set of actuators it is necessary to make sure that whatever distribution pattern he chooses the system is controllable. The concept of controllability in modern control theory

is a binary concept, either a system is controllable or it is uncontrollable. Starting from a set of actuator locations which produce an uncontrollable system, but for which the number of actuators is sufficient to produce controllability, it will usually be the case that moving one of the actuators by a distance  $\epsilon > 0$  can produce a controllable system, no matter how small the  $\epsilon$ . One expects that for a small  $\epsilon$ , even though technically the system is controllable, in some sense it will not be very controllable. It then seems natural to ask "how controllable is the system with a particular actuator distribution?" It is, therefore, reasonable to seek extensions of the established definition of controllability so as to permit a precise definition of the "degree of controllability" which would prove useful for actuator placement.

It is the purpose of this work to generate, starting from basic physical considerations, a rational definition of the degree of controllability. The definition obtained is certainly not the only possible definition, but it does have the advantage over a definition based on singular value decomposition that the physical reality of actuator saturation limitations is included in a fundamental way.

The definition is then applied to the actuator placement problem for Flexible spacecraft. With this tool the control system designer can rank the desirability of various candidate actuator distributions, and thus he would have a systematic way of picking which distribution to use.

ORIGINAL PAGE IS  
OF POOR QUALITY

## 2.2 Definition of the Degree of Controllability

Let us consider any general linear time invariant system in state variable form

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t) \quad (2.2-1)$$

where  $x^* \in R^n$  and  $u^* \in R^m$ . It should be noted that although we focus our attention on this system, the degree of controllability definition which we adopt is also applicable to more general systems of the form  $\dot{x}^*(t) = f(x^*(t), u^*(t), t)$  having a solution  $x^*(t) = 0$  ( $f(0, 0, t) = 0$ ).

Several candidate definitions were scrutinized in the course of the search for a suitable definition of the degree of controllability. It is instructive to discuss some of these candidate definitions which were considered and discarded--- the process of starting with a blind attempt at a definition and progressing to a well formulated concept highlights the characteristics that a workable definition must have. We discuss these candidate definitions in the following:

### 1. Eigenvalues of $QQ^T$ †

It is tempting to try to connect the degree of controllability to properties of the standard controllability matrix  $Q = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ , and define degree of controllability as the square root of the minimum or maximum eigenvalue of  $QQ^T$ . Five apparent

† The superscript T is used to denote transpose of a matrix or vector throughout this text.

difficulties with this definition must somehow be handled before the definition becomes viable. These are: 1) The degree of controllability is affected by a transformation of coordinates (since the eigenvalues of  $QQ^T$  are not invariant under changes in state variable representation). 2) Although this candidate definition satisfies the basic requirement that the degree of controllability is zero when the system is uncontrollable, it is not immediately clear what other physical meaning can be attached to  $QQ^T$  and hence to the size of its eigenvalues. 3) It is not clear how the stability of the system is reflected in this definition. It is much easier to control a stable system with the objective of returning the system toward the origin ( $x^*=0$  solution) than an unstable system with the same objective. (The Reverse is true when the objective is to reach out from the origin.) 4) The candidate definition does not involve a dependence on the amount of time  $T$  allotted to accomplish the control task. It can be much easier to control the system state in some directions in the state space at one time than at another time, so the degree of controllability should depend on  $T$ . 5) It is not clear that the amount of control effort needed to accomplish the control task is reflected in this definition. In the satellite described in section 2.1 where one actuator has been moved by a small amount  $\epsilon$  to produce controllability, one expects the "weak controllability" of the system to be manifested in the need for very large control actions to accomplish certain small changes in the state, and hence the control effort required is of fundamental importance in making a definition.

## 2. Impulse response

It is clear that some type of limitation or standardization of the control effort must be included in the definition. Consider a standardization which restricts the control to a unit impulse, and consider systems with  $A$  in diagonal form and with  $u^*$  a scalar. For distinct eigenvalues the system is controllable if none of the elements  $b_i$  of the column matrix  $B$  are zero. Furthermore, these components indicate how far a unit impulse control will move each state component instantaneously (impulse response), so one might suggest the  $\min |b_i|$  as a degree of controllability. Here we are trying to generalize a second standard test for controllability to obtain a degree of controllability definition. The difficulties with this candidate definition are: 1) The control actions are so restricted that the components of the state cannot be affected independently. The control of all states by a single control  $u^*$  relies on the differences in the dynamic behaviors of the states. 2) The candidate definition does not involve a dependence on the eigenvalues of the system, which means the information regarding the stability of the system is absent. The rest of the argument is the same as in (3) of candidate definition 1. 3) The candidate definition does not involve a dependence on  $T$  and exactly the same argument as in (4) of candidate definition 1 holds.

## 3. Energy due to an impulse

When an impulse is applied to a physical system there occurs a change in the total energy of the system, and one might wonder if this change in energy due to an impulse could not be used to define the degree of controllability. Since this is dependent on the impulse input

11

it will suffer from all the drawbacks of the impulse response discussed under the candidate definition 2.

These candidate definitions do not seem to include the effects of all pertinent variables. Hence, it will be necessary to build the definition from more fundamental considerations. It is interesting to note that, in certain special cases, the resulting degree of controllability definition will be a modified version of the candidate definition 2 (and by employing a different approach involving singular value decomposition of matrices something of the general form of the first candidate definition can result).

It is now evident that the definition of the degree of controllability, besides being in some sense a measure of how easy it is for the controller to control the system, must in some way handle five things:

- 1) It must have the property that the degree of controllability is zero when the system is uncontrollable.
- 2) It must reflect in some way the stability information of the system.
- 3) It must somehow consider dependence on total time  $T$ .
- 4) It must standardize the control effort in some way.
- 5) The control objective must be restricted.

Concerning the last point, certainly different control objectives should influence the choice of the control system design, and hence the degree of controllability of a candidate design should be keyed to the objective involved. In a large class of problems (regulator problems), the equilibrium solution  $x^*=0$  to equation (2.2-1) is of primary

ORIGINAL PAGE IS  
OF 200

importance, and the control objective is to return  $x^*$  to zero after a disturbance. Since this is the most common attitude and shape control problem for flexible spacecraft, we will restrict ourselves to this objective. Concerning the standardization of the control effort we will require that the control components satisfy  $|u_i| \leq 1$  for  $i=1,2,\dots,m$ , which represents realistic physical limitations of the actuator capabilities. Note that the use of one as the bound for all control components implies normalizing each component of  $u^*$  to produce a new control vector  $u$ , and adjusting the matrix  $A$  to produce a new matrix  $B$ .

Controllability requires the existence of a control function which can transfer any initial state to any final state in finite time. With our more limited control objective, the degree of controllability should be related to the volume of initial system states (or states resulting from disturbances) which can be returned to the desired state  $x^*=0$  in time  $T$  using the bounded controls. Consider the nature of this volume in more detail. In a controllable system, more initial states in any direction can be returned to the origin if the system is stable than if it were unstable. Hence, the volume of initial system states is greater in the case of a stable controllable system than in the case of an unstable controllable system. Now, in an uncontrollable system there will be at least one direction in the state space for which initial conditions in this direction cannot be returned to the origin, and the volume will lose one or more dimensions. For a controllable system whose parameters are such that it is nearly uncontrollable, only initial conditions very close to  $x^*=0$  along the above mentioned direction could



be returned to the origin in time  $T$  using the bounded controls. Hence, we will generate a definition of the degree of controllability based on the minimum distance from the origin to a normalized state that cannot be brought to the origin in time  $T$ . More loosely it is the minimum disturbance from which the system cannot recover in time  $T$ .

The coordinates of a state space will very rarely all have the same physical units, and hence it is clear that comparing distances in the state space will require that each coordinate must be made unitless by normalization. How should one choose the normalization to use? Recognize that when comparing two controller designs for controlling the same dynamic system, the needed minimum distance for each design will usually correspond to a different direction in state space. Hence, ranking of the degree of controllability of the two systems will depend on comparison of distances in different directions, and this implies that we must be equally interested in controlling deviations of the state from  $x^* = 0$  in all directions in the state space. In order to accomplish this the control system designer must specify  $n-1$  numbers which represent his degree of interest in controlling each component of the state. This could be done, for example, by determining the deviations of  $x_1^*$ ,  $x_2^*$ , ...,  $x_{n-1}^*$ , which would be considered of equal importance to a deviation of  $x_n^* = 1$ . The reciprocals of these numbers would then be used to produce normalization factors for each of the coordinates of the state space giving a new state vector  $x$ . The system equations expressed in terms of the normalized state  $x$  and normalized control  $u$  are then written as

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.2-2)$$

$$|u_i| \leq 1 \quad i=1,2,\dots,m$$

Just as in optimal control theory where the control system designer must be specific about his goal by specifying a cost functional, in order to define the degree of controllability, it is necessary to be fully specific not only about the objective of keeping  $x=0$ , but also about the relative importance of keeping each component of  $x$  near zero.

Relative to the normalized system (2.2-2) we are now ready to make the following definitions:

Definition 2.1: The recovery region for time  $T$  for normalized system (2.2-2) is the set

$$\mathcal{R} = \{ x(0) \mid \exists u(t), t \in [0, T], |u_i(t)| \leq 1 \text{ for } i=1,2,\dots,m \text{ and } x(T) = 0 \}$$

Definition 2.2: The degree of controllability in time  $T$  of the  $x=0$  solution of normalized system (2.2-2) is defined as

$$\rho = \inf \| x(0) \| \quad \text{for } x(0) \notin \mathcal{R}$$

where  $\| \cdot \|$  represents the Euclidean norm.

Thus, the recovery region identifies all of the initial conditions (or disturbed states) which can be returned to the origin in time  $T$  using the bounded controls. And the degree of controllability is a

scalar measure of the size of the region, where the scalar is chosen as the shortest distance from the origin to an initial state which cannot be returned to the origin in time  $T$ .

The degree of controllability, as defined, is keyed to the state vector  $x$  employed. No transformations of coordinates can be allowed once the normalization has been specified (unless the norm used in the definition is adjusted to compensate for the resulting distortion of the state space).

It should be pointed out that although Definition 2.2 incorporates all the properties which were identified as necessary in the definition of the degree of controllability, it is not necessarily unique in doing so. For example, a standardization of the control effort in terms of energy can also be employed, but the inequality saturation constraints on the controls used here represents the more realistic situation.

ORIGINAL PAGE IS  
OF FOUR PAGES

### 2.3 Concepts for Approximating the Recovery Region

In order to make the definition of the degree of controllability useful, it is necessary to develop a simple algorithm to generate at least an approximation to the distance  $\rho$ . This necessitates approximating the recovery region  $\mathcal{R}$ .

Note that the solution of (2.2-2) is given by (see [2], [3])

$$x(t) = \Phi(t, t_0)x(t_0) + \Phi(t, t_0) \int_{t_0}^t \Phi(t_0, s)Bu(s) ds \quad (2.3-1)$$

where  $\Phi(t, t_0)$  is the state transition matrix for (2.2-2). A complete explanation of  $\Phi$  can be found in [2]. Without any loss of generality we can assume the initial time  $t_0=0$  and the final time  $t=T$ . Also we will denote  $x(0)$  by  $x_0$  and  $x(T)$  by  $x_T$ . The solution (2.3-1) then is simply

$$x_T = \Phi(T, 0)x_0 + \Phi(T, 0) \int_0^T \Phi(0, t)Bu(t) dt \quad (2.3-2)$$

The displacement in the state space in time  $T$  is  $x_T - x_0$ , and let

$$\delta^* = x_T - x_0 \quad (2.3-3)$$

and the distance moved during time  $T$  is the Euclidean norm  $\|\delta^*\|$ . From (2.3-2) the displacement  $\delta^*$  can be expressed in either of the following form:

$$\delta_1^* = [E - \Phi^{-1}(T, 0)]x_T + \int_0^T \Phi(0, t)Bu(t) dt \quad (2.3-4)$$

or

$$\delta_2^* = [\Phi(T,0) - E]x_0 + \Phi(T,0) \int_0^T \Phi(0,t)Bu(t) dt \quad (2.3-5)$$

where  $E$  is the unit matrix and the subscripts on  $\delta^*$  are used to identify the two forms. In (2.3-4) the displacement is expressed in terms of the final state  $x_T$  and in (2.3-5) it is expressed in terms of the initial state  $x_0$ . Our interest lies in  $\delta_1^*$ , in which  $x_T=0$  and we termed the set of states  $x_0$  as a recovery region  $\mathcal{R}$  (see Def.(2.1)). A companion region  $\mathcal{R}$  which is the set of states  $x_T$  for which  $x_0=0$  can be defined in a similar way to the recovery region  $\mathcal{R}$ . The region  $\mathcal{R}$  then represents the reachable states from the origin. Putting  $x_T=0$  for  $\delta_1^*$ , and  $x_0=0$  for  $\delta_2^*$  in (2.3-4) and (2.3-5), respectively, we can obtain

$$\delta_1 = \int_0^T \Phi(0,t)Bu(t) dt \quad (2.3-6)$$

and

$$\delta_2 = \Phi(T,0) \int_0^T \Phi(0,t)Bu(t) dt \quad (2.3-7)$$

where the (\*) on  $\delta_i$  ( $i=1,2$ ) has been removed to indicate that the displacement is about the origin ( $x_T$  or  $x_0$  is zero).

Note that for linear time invariant systems the fundamental matrix can be written as

$$\Phi(t,\tau) = e^{A(t-\tau)} \quad (2.3-3)$$

from which

$$\Phi(\tau, 0) = e^{A\tau} ; \quad \Phi(0, t) = e^{-At} \quad (2.3-9)$$

Two remarks are worth making regarding the comparisons between the regions  $\mathcal{R}$  and  $\mathcal{N}$  corresponding to the displacements  $\delta_1$  and  $\delta_2$  when identical control  $u(t)$  is applied during the same time  $T$  for both cases.

1. The region  $\mathcal{N}$  of the reachable states from the origin in time  $T$  is the same as the recovery region  $\mathcal{R}$  if the control process were taking place in a time domain where time runs backward (interchange  $t_0 (=0)$  and  $T$  in  $\delta_2$  and obtain the same magnitude as of  $\delta_1$  ).
2. The region  $\mathcal{N}$  is larger than the region  $\mathcal{R}$  for time  $T$  if the system is unstable about the origin (i.e., the real part of at least one eigenvalue of  $A$  is greater than zero). This is because there is always at least one component of  $\delta_2$  whose magnitude is greater than the corresponding component of  $\delta_1$ , (due to the influence of the exponential terms,  $e^{\lambda_i t}$ , in  $\Phi(T, 0)$ ). If the system is stable the converse is true.

In the following analysis we will be concerned only with the recovery region  $\mathcal{R}$  and the corresponding equation for the state space displacement is (2.3-5)<sup>†</sup>.

<sup>†</sup>The superscript 1 will be omitted henceforth and  $\delta$ , unless otherwise mentioned, corresponds to the recovery region  $\mathcal{R}$ , i.e., in (2.3-5).

By the Caley-Hamilton theorem the state transition matrix can be written as

$$\Phi(0,t) = e^{-At} = \sum_{\alpha=0}^{n-1} \psi_{\alpha}(\cdot) A^{\alpha} \quad (2.3-10)$$

where the  $\psi_{\alpha}$  are scalar functions of time. Partition the B matrix into column matrices  $b_j$ , and define the following matrices

$$B = [b_1 \ b_2 \ b_3 \ \dots \ b_m] \quad (2.3-11)$$

$$\Psi^T = [\psi_0 \ \psi_1 \ \psi_2 \ \dots \ \psi_{n-1}] \quad (2.3-12)$$

$$Q = [B \ AB \ A^2B \ \dots \ A^{n-1}B] \quad (2.3-13)$$

$$Q_p = [b_p \ Ab_p \ A^2b_p \ \dots \ A^{n-1}b_p] \quad (2.3-14)$$

Then  $\delta$  can be represented in the following alternative forms

$$\delta = \sum_{\alpha=0}^{n-1} \sum_{p=1}^m \left\{ \int_0^T \psi_{\alpha}(t) u_p(t) dt \right\} A^{\alpha} b_p \quad (2.3-15)$$

$$= \sum_{p=1}^m \int_0^T [\psi_0 b_p + \psi_1 Ab_p + \dots + \psi_{n-1} A^{n-1} b_p] u_p dt \quad (2.3-16)$$

$$= \sum_{p=1}^m \int_0^T [Q_p \Psi] u_p dt \quad (2.3-17)$$

For the purposes of illustrating certain concepts, let us restrict

ourselves to the case of a scalar control so that the summations over  $\beta$  as well as the  $\beta$  subscripts in the above can be dropped, and  $B$  is a column matrix  $b$ . Also let  $n=2$  for simplicity. Suppose the recovery region is as shown by Region I in Fig. 1. The maximum  $x_1$  component of any state in the recovery region is obtained by using the control  $u$  equal to minus the signum function of the first component of the vector  $[q\psi]$  in (2.3-17), since this maximizes the  $x_1$  component of the integrand at each time  $t$ . The right hand side (and left hand side) of the rectangle enclosing this recovery region in Fig. 1 can thus be found by integrating the first component in (2.3-17) using this control. If desired the point at which the recovery region touches this side is obtained by integrating the second component of (2.3-17) using this control. The top and bottom of the rectangle are found similarly.

The rectangle obtained in this manner might be considered an approximation to the recovery region, and then the shortest distance from the origin to one of the sides might be considered an approximation,  $\hat{\rho}$ , to the degree of controllability,  $\rho = \rho_I$ . Note that this necessarily produces a  $\hat{\rho}$  which is an upper bound for the degree of controllability. In some cases this approximation is a tight one, but often it is not. Suppose the recovery region was Region II of Fig. 1. This corresponds to a system which has a much poorer degree of controllability,  $\rho = \rho_I$ , yet the approximation  $\hat{\rho}$  remains the same. In fact, suppose that  $\rho_I \rightarrow 0$  in such a way that Region II degenerates to a line forming a diagonal of the rectangle in Fig. 1. Then the system is an uncontrollable system, but the approximation  $\hat{\rho}$  still predicts a good degree of controllability. Hence, this approximation must be



rejected.

For the case of a scalar control being considered, this shortcoming can be eliminated by using  $\delta$  as expressed in (2.3-15) and maximizing components along  $A^\alpha b$ . The control

$$u(t) = -\text{sgn}[\psi_\alpha(t)] \quad (2.3-18)$$

extremizes the coefficient of the vector  $A^\alpha b$  in (2.3-2). It will simultaneously produce some components along the other vectors  $A^\gamma b$  for  $\gamma \neq \alpha$ . This is a maximization of a component of the vector  $\delta$  but it is a component as seen in a nonorthogonal set of coordinates. Hence, the upper bounds obtained in the various directions define a parallelogram (more generally an  $n$  dimensional parallelepiped) which can be considered as an approximation to the recovery region, as shown in Fig. 2. As before there is some point on each side of the parallelogram which is in the recovery region, but no point outside the parallelogram is in the region.

The minimum distance to a side of the parallelogram, i.e., the minimum perpendicular distance to a side, is an approximation  $\rho^*$  to the degree of controllability,  $\rho$ . When the system becomes uncontrollable, the columns of  $Q$  become linearly dependent, and hence the perpendicular distance to one of the sides becomes zero. This means that this  $\rho^*$  has the essential property that  $\rho^* = 0$  whenever  $\rho = 0$ .

We conclude that for the scalar control case we have a viable method of approximating the degree of controllability. A simple method will be presented in a later section to determine the needed minimum

perpendicular distance.

This approximation is still an upper bound, and it can be improved, in fact made arbitrarily good, by considering more directions in the state space. Let  $e$  be any desired unit vector expressed as a column matrix of components. By examining (2.3-17) the state in the recovery region having a maximum component along the direction  $e$  is obtained using the control

$$u = -\operatorname{sgn} [e^T q \psi] \quad (2.3-19)$$

and hence no points in the recovery region lie beyond the line perpendicular to  $e$  and a distance

$$\int_0^T |e^T q \psi| dt \quad (2.3-20)$$

from the origin (but at least one point in the recovery region lies on the line). Figure 3 illustrates how use of three  $e$ 's ( $e_I$ ,  $e_{II}$ , and  $e_{III}$ ) identifies three tangents to the recovery region, and when taken together they begin to approximate the region boundary. Let  $\tilde{\rho}$  be the minimum value of (2.3-20) for any set of directions  $e$  considered. Then an improved estimate of the degree of controllability is  $\rho^{**} = \min(\rho^*, \tilde{\rho})$ , and  $\rho^{**} > \rho$  can be made arbitrarily close to the true degree of controllability  $\rho$  by picking a sufficient number of directions  $e$ . This method of improving the approximation to the degree of controllability will also be generalized to the multiple control case.

## 2.4 Fundamental Equation for Recovery Region Approximation in the Multiple Control Case

The previous section presented a procedure for generating an approximation  $\rho^*$  to the degree of controllability  $\rho$  in the case of a scalar control  $u$ . The procedure required the use of  $n$  carefully chosen directions in the state space,  $b, Ab, \dots, A^{n-1}b$ , in the approximation to the recovery region in order to insure that  $\rho^*$  had the property that  $\rho^* = 0$  if and only if the system is uncontrollable. If the control vector is  $m$  dimensional with  $m > 1$  it is no longer obvious how to obtain this property, since the columns of the  $Q$  matrix necessarily contain linearly dependent vectors. Instead, we will consider the eigenvectors and generalized eigenvectors of the  $A$  matrix of (2.2-2) as the  $n$  linearly independent directions in the state space. Certainly some modifications must apply when these vectors are complex. In the single control case the value of  $\rho^*$  became zero when the system became uncontrollable because linear dependence of the vectors  $b, Ab, \dots, A^{n-1}b$  implies the collapse of at least one dimension of the parallelepiped. The vectors chosen here for the multi-dimensional control case do not exhibit this reduction. Nevertheless, it will be shown in the next section that the desired property of the resulting  $\rho^*$  can be demonstrated under fairly general assumptions. This section is devoted to generating the appropriate expression for  $\delta$  equivalent to equations (2.3-6, 2.3-15-2.3-17), expressed in terms of components in these eigenvector directions.

Let  $J$  be the Jordan canonical form of the matrix  $A$ , and let  $P$  be

ORIGINAL PAGE 1  
OF POOR QUALITY

the matrix of eigenvectors and generalized eigenvectors so that

$$P^{-1}AP = J ; \quad J = \text{diag} [ J_1, J_2, \dots, J_r ] \quad (2.4-1)$$

where the  $J_k$  are the square Jordan blocks of dimension  $\nu_k$ ,  $k=1,2,\dots,r$ . A diagonal Jordan block is of order one. Associated with each Jordan block is an eigenvalue  $\lambda_k$  so that  $r \leq n$  and  $r$  is greater than or equal to the number of distinct eigenvalues. Also, let  $p_j$  be the  $n$  columns of  $P$ , and  $\hat{p}_j^T$  be the  $n$  rows of  $P$  (the left eigenvectors and generalized left eigenvectors). Every Jordan block  $J_k$  corresponds to one independent eigenvector of  $A$  and  $\nu_k - 1$  generalized eigenvectors, and the same is true for the left eigenvectors and generalized left eigenvectors.

It is appropriate to list here a few results on matrix algebra (see Appendix A for derivation) which we will use in our further analysis:

$$P^{-1}e^{-At} = e^{-Jt} P^{-1}$$

$$e^{-Jt} = \text{diag} [ e^{-J_1 t}, e^{-J_2 t}, \dots, e^{-J_r t} ]$$

$$e^{-J_k t} = e^{-\lambda_k t} e^{-\hat{N}_k t} \quad (2.4-2)$$

where

$$e^{-\hat{N}_k t} = \begin{bmatrix} 1 & -t & \frac{(-t)^2}{2!} & \frac{(-t)^3}{3!} & \dots & \frac{(-t)^{j_k-1}}{(j_k-1)!} \\ 0 & 1 & -t & \frac{(-t)^2}{2!} & \dots & \frac{(-t)^{j_k-2}}{(j_k-2)!} \\ 0 & 0 & 1 & -t & \dots & \frac{(-t)^{j_k-3}}{(j_k-3)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots 1 \dots & -t \\ 0 & 0 & 0 & 0 & \dots 0 \dots & 1 \end{bmatrix} \quad (2.4-3)$$

$$k = 1, 2, 3, \dots, r, \quad r \leq n$$

With these we are ready to derive the desired fundamental equation for  $\delta$ , the state space displacement. The form for this fundamental equation is defined in the following:

Definition 2.3: A spatial form for the state space displacement is defined as the form

$$\delta = C \int_0^T \alpha(t) dt$$

in which  $C$  is a set of  $l$  ( $l \leq n$ ) time invariant  $n \times 1$  column vectors  $c_1, c_2, \dots, c_l$ , in the real  $n$  dimensional space, and  $\alpha(t)$  is a set of  $l$  time dependent real scalar elements  $\alpha_1(t), \alpha_2(t), \dots, \alpha_l(t)$ , so that one element of  $\alpha(t)$  is associated with every column vector of  $C$ . In matrix notation, if

$$C = [c_1 \ c_2 \ \dots \ c_l] \quad ; \quad \alpha^T(t) = [\alpha_1(t) \ \alpha_2(t) \ \dots \ \alpha_l(t)]$$

then

$$\begin{aligned}\delta &= [c_1 \ c_2 \ \dots \ c_l] \int_0^T \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_l(t) \end{bmatrix} dt \\ &= c_1 \int_0^T \alpha_1(t) dt + c_2 \int_0^T \alpha_2(t) dt + \dots + c_l \int_0^T \alpha_l(t) dt\end{aligned}$$

#### 2.4.1 Reduction of $\delta$ -Equation to the Spatial Form

The  $\delta$ -equation from (2.3-6) using (2.3-9) is

$$\delta = \int_0^T e^{-At} B u(t) dt \quad (2.4-4)$$

Premultiplying the integrand on the right side by a unit matrix  $E = PP^{-1}$ , where  $P$  is the transformation matrix of  $A$  for the Jordan canonical form (see (2.4-1)),

$$\delta = \int_0^T PP^{-1} e^{-At} B u(t) dt$$

Substituting for  $P^{-1}e^{-At}$  from (2.4-2)

$$\delta = \int_0^T P e^{-Jt} P^{-1} B u(t) dt \quad (2.4-5)$$

It is convenient to partition the  $n \times n$  matrices  $P$ ,  $P^{-1}$ , column wise and row wise, respectively. Let  $\tilde{P}_k$ ,  $k=1,2,3,\dots,r$  denote a typical

partition of  $P$  consisting of  $\nu_k$  columns of  $P$ , and let  $\hat{P}_k$  denote a typical partition of  $P^{-1}$  consisting of  $\nu_k$  rows of  $P^{-1}$ . Also, it is easier to identify the column vectors  $p_j$  (or the row vectors  $\hat{p}_j^T$ ) in relation to the partition to which they belong as well as to the global matrix  $P$  (or  $P^{-1}$ ) if we adopt a modified subscript notation. So let  $p_{L^{k-1}+i}$ ,  $\hat{p}_{L^{k-1}+i}^T$ ,  $k=1,2,3,\dots,r$ ,  $i=1,2,3,\dots,\nu_k$ , denote the  $p_j$ ,  $\hat{p}_j^T$  vectors where

$$L^k = \begin{cases} \sum_{l=1}^k \nu_l & , \quad k=1,2,\dots,r \\ 0 & , \quad k=0 \end{cases} \quad (2.4-6)$$

The index  $k$  refers to the partition  $\tilde{P}_k$ ,  $\hat{P}_k$  and  $i$  refers to the location in a partition (e.g.,  $k=2, i=4$  refers to the 4th column in  $\tilde{P}_2$  and  $(L^1+4)$ th column in  $P$ . The same is true for row vectors of  $P^{-1}$ ). Also, note that  $L^{k-1} + \nu_k = L^k$ . The  $L^k$ th row vector of  $P^{-1}$ ,  $k=1,2,\dots,r$ , is of great significance in our analysis and we will refer to it often. So it is expedient to define the following:

Definition 2.4: The column vectors  $\hat{p}_k$ ,  $L^k = \sum_{j=1}^k \nu_j$ ,  $k=1,2,\dots,r$ , which are the transpose of the row vectors  $\hat{p}_{L^k}^T$  of  $P^{-1}$  are defined as foundation vectors.

We give this name to these vectors because these are the (left) eigenvectors on which the chain of generalized eigenvectors is built corresponding to each Jordan block.

With these notations

$$\tilde{P}_k = \begin{bmatrix} p_{L^{k-1}+1} & p_{L^{k-1}+2} & p_{L^{k-1}+3} & \dots & p_{L^{k-1}+\nu_k} \end{bmatrix}_{n \times \nu_k}$$

$$\hat{P}_k = \begin{bmatrix} \hat{p}_{L+1}^T \\ \hat{p}_{L+2}^T \\ \vdots \\ \hat{p}_{L+k}^T \end{bmatrix} \quad (2.4-7)$$

$$P = [\tilde{P}_1 \tilde{P}_2 \tilde{P}_3 \dots \tilde{P}_r]; \quad P^{-1} = \begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \vdots \\ \hat{P}_r \end{bmatrix} \quad (2.4-8)$$

Therefore, in partitioned form the integrand of (2.4-5) using (2.4-2) is

$$P e^{-Jt} P^{-1} B u = [\tilde{P}_1 \tilde{P}_2 \dots \tilde{P}_r] \begin{bmatrix} e^{-J_1 t} & & 0 \\ & e^{-J_2 t} & \\ 0 & \ddots & e^{-J_r t} \end{bmatrix} \begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \vdots \\ \hat{P}_r \end{bmatrix} B u \quad (2.4-9)$$

$$P e^{-Jt} P^{-1} B u = [\tilde{P}_1 \tilde{P}_2 \dots \tilde{P}_r] \begin{bmatrix} e^{-J_1 t} \hat{P}_1 B u \\ e^{-J_2 t} \hat{P}_2 B u \\ \vdots \\ e^{-J_r t} \hat{P}_r B u \end{bmatrix} \quad (2.4-10)$$



In the above the mathematical quantities  $\tilde{p}_k$ ,  $\hat{p}_k$ ,  $J_k$  could in general be complex due to complex eigenvalues of the system. But our system  $(x, u, A, B)$  is real and so is the displacement  $\delta$ . In order to be able to attach real meaning to our results the recovery region must be constructed in real space.

#### 2.4.2 Reduction of the $\delta$ -equation to Real Domain

Because our system is real we know that the eigenvalues are either real or occur in complex conjugate pairs. For real eigenvalues we will choose real eigenvectors and for pairs of complex conjugate eigenvalues we will choose eigenvectors in conjugate pairs; i.e., if  $p_i$  is an eigenvector of  $A$  corresponding to an eigenvalue  $\lambda_i$ , then  $\bar{p}_i^\dagger$  is an eigenvector corresponding to the eigenvalue  $\bar{\lambda}_i$ , because

$$A p_i = \lambda_i p_i ; \quad \overline{A p_i} = \overline{\lambda_i p_i} , \text{ or } A \bar{p}_i = \bar{\lambda}_i \bar{p}_i \quad (2.4-11)$$

Similarly, for left eigenvectors the conjugate property is true, i.e.,

$$\hat{p}_i^T A = \lambda_i \hat{p}_i^T ; \quad \overline{\hat{p}_i^T A} = \overline{\lambda_i \hat{p}_i^T} , \text{ or } \hat{p}_i^T A = \bar{\lambda}_i \bar{\hat{p}_i^T} \quad (2.4-12)$$

For real eigenvalues the left eigenvectors are also real. The conjugate property by extension is true for all generalized eigenvectors (left or

<sup>†</sup> Throughout this text a bar will be used to denote the complex conjugate of a mathematical object.

right).

In (2.4-10) the right side is a product of two matrices each in partitioned form according to the order of  $J_k$ . The location per se of any partition in a matrix does not matter so long as the partitions in the two matrices correspond to each other (e.g.,  $\tilde{P}_k$  and  $e^{-J_k^t \hat{P}_k B u}$ ). So we can assume that in (2.4-10) the partitions  $e^{-J_k^t \hat{P}_k B u}$  in the second matrix are arranged such that those corresponding to real eigenvalues (real  $J_k$ ) are on top, followed by pairs of partitions, each pair corresponding to a pair of complex conjugate eigenvalues (complex  $J_k$ ). And the partitions  $\tilde{P}_k$  in the first matrix correspond to those in the second matrix.

We will let  $\mathcal{C}$  denote the class of all real quantities, and  $\mathcal{C}^*$  denote the class of all complex quantities. For class  $\mathcal{C}$  the partitions  $\tilde{P}_k$ ,  $e^{-J_k^t \hat{P}_k B u}$  are real, and for class  $\mathcal{C}^*$  they are complex.

It is now convenient to write (2.4-10) as follows:

$$P e^{-J^t P^{-1} B u} = \begin{bmatrix} \mathcal{C}_I & \mathcal{C}_I^* \end{bmatrix} \begin{bmatrix} \mathcal{C}_I \\ \mathcal{C}_I^* \end{bmatrix} \quad (2.4-13)$$

where

$\mathcal{C}_I$  is a set of real partitions  $\{\tilde{P}_k\}$ ,  $\lambda_k \in \mathcal{C}$

$\mathcal{C}_I^*$  is a set of complex partitions  $\{\tilde{P}_k\}$ ,  $\lambda_k \in \mathcal{C}^*$

$\mathcal{C}_I$  is a set of real partitions  $\{e^{-J_k^t \hat{P}_k B u}\}$ ,  $\lambda_k \in \mathcal{C}$

$\mathcal{C}_I^*$  is a set of complex partitions  $\{e^{-J_k^t \hat{P}_k B u}\}$ ,  $\lambda_k \in \mathcal{C}^*$

Now, we can formulate the reduction of  $\delta$  to the spatial form in real

domain by means of the following theorem.

Theorem 2.1: For any linear time invariant system (2.2-2) the displacement  $\delta$  in the state space can always be reduced to the spatial form defined in Def.(2.3) in which  $C, \alpha(t)$  are real valued.

Proof: Consider the equation (2.4-13). But for the sets  $C_I^*, C_{II}^*$  this spatial form of  $\delta$  would have been in real space. Now, consider the expansion of (2.4-13) in terms of real and complex classes, i.e.,

$$P e^{-Jt} P^{-1} B u = C_I C_{II} + C_I^* C_{II}^* \quad (2.4-14)$$

It is therefore necessary to express  $C_I^* C_{II}^*$  in real terms to obtain the real spatial form for  $\delta$ . Expand  $C_I^* C_{II}^*$  partition wise and group the resulting partition matrix products in pairs according to the complex conjugate eigenvalues; i.e., consider pairs like

$$\tilde{P}_j e^{-J_j t} \hat{P}_j B u + \tilde{P}_{j+1} e^{-J_{j+1} t} \hat{P}_{j+1} B u \quad (2.4-15)$$

(which correspond to the pairs of eigenvalues  $\lambda_j, \bar{\lambda}_j$ ). Note that the order  $\nu_j$  of  $J_j$  and  $\nu_{j+1}$  of  $J_{j+1}$  are equal ( $\nu_{j+1} = \nu_j$ ), because of the conjugate property of the eigenvalue  $\lambda_j$  and eigenvector  $\hat{p}_j$ , etc.. Furthermore, from this conjugate property (see (2.4-11), (2.4-12)) we have the following:

$$J_{j+1} = \bar{J}_j ; \quad \tilde{P}_{j+1} = \bar{\tilde{P}}_j ; \quad \hat{P}_{j+1} = \bar{\hat{P}}_j \quad (2.4-16)$$

and hence

$$\begin{aligned}\tilde{p}_{j+1} e^{-\tilde{J}_{j+1}t} \hat{p}_{j+1} B u &= \overline{\tilde{p}_j e^{-\tilde{J}_j t} \tilde{p}_j B u} \\ &= \overline{(\tilde{p}_j e^{-\tilde{J}_j t} \hat{p}_j B u)}\end{aligned}\quad (2.4-17)$$

Thus, the second term in (2.4-15) is the complex conjugate of the first term; therefore, (2.4-15) becomes

$$\tilde{p}_j e^{-\tilde{J}_j t} \hat{p}_j B u + \tilde{p}_{j+1} e^{-\tilde{J}_{j+1} t} \hat{p}_{j+1} B u = 2 \operatorname{Re} [\tilde{p}_j e^{-\tilde{J}_j t} \hat{p}_j B u] \quad (2.4-18)$$

where  $\operatorname{Re}[\cdot]$  denotes the real part of a quantity. We now proceed by separating the real and imaginary parts as shown below. Let

$$\lambda_j = -\zeta_j + i \omega_j ; \quad \tilde{p}_j = \tilde{p}_j^R + i \tilde{p}_j^I ; \quad \hat{p}_j = \hat{p}_j^R + i \hat{p}_j^I \quad (2.4-19)$$

where the superscripts R, I stand for the real and imaginary parts, respectively, of the quantity. Substituting for  $\lambda_j$  from (2.4-19) in (2.4-2)

$$\begin{aligned}e^{-\tilde{J}_j t} &= e^{-\lambda_j t} e^{-\hat{N}_j t} = e^{-(\zeta_j + i \omega_j) t} e^{-\hat{N}_j t} \\ &= e^{\zeta_j t} e^{-i \omega_j t} e^{-\hat{N}_j t} \\ &= e^{\zeta_j t} e^{-\hat{N}_j t} (\cos \omega_j t - i \sin \omega_j t) \\ &= (e^{\zeta_j t} e^{-\hat{N}_j t} \cos \omega_j t) - i (e^{\zeta_j t} e^{-\hat{N}_j t} \sin \omega_j t)\end{aligned}\quad (2.4-20)$$

Using (2.4-19) and (2.4-20) on the right side of (2.4-18),

$$\begin{aligned}
 2 \operatorname{Re} [ \tilde{p}_j e^{-j_j t} \hat{p}_j B u ] &= 2 \operatorname{Re} \{ ( \tilde{p}_j^R + i \tilde{p}_j^I ) [ ( e^{j_j t} e^{-\hat{N}_j t} \cos \omega_j t ) \\
 &\quad - i ( e^{j_j t} e^{-\hat{N}_j t} \sin \omega_j t ) ] ( \hat{p}_j^R + i \hat{p}_j^I ) B u \} \\
 &= 2 \operatorname{Re} \{ [ \tilde{p}_j^R e^{j_j t} e^{-\hat{N}_j t} ( \hat{p}_j^R B u \cos \omega_j t + \hat{p}_j^I B u \sin \omega_j t ) \\
 &\quad + \tilde{p}_j^I e^{j_j t} e^{-\hat{N}_j t} ( \hat{p}_j^R B u \sin \omega_j t - \hat{p}_j^I B u \cos \omega_j t ) ] \\
 &\quad + i [ \text{imaginary part} ] \} \\
 &= 2 \tilde{p}_j^R e^{j_j t} e^{-\hat{N}_j t} ( \hat{p}_j^R B u \cos \omega_j t + \hat{p}_j^I B u \sin \omega_j t ) \\
 &\quad + 2 \tilde{p}_j^I e^{j_j t} e^{-\hat{N}_j t} ( \hat{p}_j^R B u \sin \omega_j t - \hat{p}_j^I B u \cos \omega_j t )
 \end{aligned}$$

(2.4-21)

This can be expressed in matrix form in terms of partitions, and hence (2.4-15) can be written as

$$\begin{bmatrix} \tilde{p}_j & \tilde{p}_{j+1} \end{bmatrix} \begin{bmatrix} e^{-j_j t} \hat{p}_j B u \\ e^{-j_{j+1} t} \hat{p}_{j+1} B u \end{bmatrix} = \begin{bmatrix} \tilde{p}_j^R & \tilde{p}_j^I \end{bmatrix} \begin{bmatrix} 2 e^{j_j t} e^{-\hat{N}_j t} ( \hat{p}_j^R B u \cos \omega_j t + \hat{p}_j^I B u \sin \omega_j t ) \\ 2 e^{j_j t} e^{-\hat{N}_j t} ( \hat{p}_j^R B u \sin \omega_j t - \hat{p}_j^I B u \cos \omega_j t ) \end{bmatrix}$$

(2.4-22)

Thus, we see that the complex partitions on the left side are simply

replaced by similar real partitions on the right side.

Proceeding in a similar manner (2.4-15) can be expressed in real form, as on the right side of (2.4-21), for all pairs of complex conjugate eigenvalues. Now, we can rebuild the matrices in partition form to replace  $C_I^*$  and  $C_{II}^*$  in (2.4-13). Denoting these replacements by  $\hat{C}_I, \hat{C}_{II}$  for  $C_I^*, C_{II}^*$ , respectively, we can write (2.4-13) as

$$\begin{aligned} P e^{-Jt} P^{-1} B u &= C_I C_{II} + C_I^* C_{II}^* \\ &= C_I C_{II} + \hat{C}_I \hat{C}_{II} \\ &= \begin{bmatrix} C_I & \hat{C}_I \end{bmatrix} \begin{bmatrix} C_{II} \\ \hat{C}_{II} \end{bmatrix} \end{aligned} \quad (2.4-23)$$

where  $C_I, C_{II}$  are sets of old partitions which were real to start with and  $\hat{C}_I, \hat{C}_{II}$  are sets of modified real partitions replacing the sets of complex partitions  $C_I^*, C_{II}^*$ . Now, the right side in (2.4-22) is completely real, and therefore using this in (2.4-5) the state space displacement, expressed in real domain, becomes

$$\begin{aligned} \delta &= \int_0^T \begin{bmatrix} C_I & \hat{C}_I \end{bmatrix} \begin{bmatrix} C_{II} \\ \hat{C}_{II} \end{bmatrix} dt \\ &= \begin{bmatrix} C_I & \hat{C}_I \end{bmatrix} \int_0^T \begin{bmatrix} C_{II} \\ \hat{C}_{II} \end{bmatrix} dt \end{aligned} \quad (2.4-24)$$

where

$$G_I = [\tilde{P}_1 \ \tilde{P}_2 \ \tilde{P}_3 \ \dots \ \tilde{P}_l]$$

$$G_I = \begin{bmatrix} e^{-J_1 t} \hat{P}_1 B u \\ e^{-J_2 t} \hat{P}_2 B u \\ \vdots \\ e^{-J_l t} \hat{P}_l B u \end{bmatrix}, \quad j=1, 2, \dots, l, \quad \lambda_j \in \mathcal{C}$$

$$\hat{G}_I = [\tilde{P}_1^R \ \tilde{P}_1^I \ \tilde{P}_2^R \ \tilde{P}_2^I \ \dots \ \tilde{P}_l^R \ \tilde{P}_l^I]$$

$$\hat{G}_{II} = \begin{bmatrix} 2 e^{\zeta_1 t} e^{-\hat{N}_1 t} (\hat{P}_1^R \cos \omega_1 t + \hat{P}_1^I \sin \omega_1 t) B u \\ 2 e^{\zeta_1 t} e^{-\hat{N}_1 t} (\hat{P}_1^R \sin \omega_1 t - \hat{P}_1^I \cos \omega_1 t) B u \\ 2 e^{\zeta_2 t} e^{-\hat{N}_2 t} (\hat{P}_2^R \cos \omega_2 t + \hat{P}_2^I \sin \omega_2 t) B u \\ 2 e^{\zeta_2 t} e^{-\hat{N}_2 t} (\hat{P}_2^R \sin \omega_2 t - \hat{P}_2^I \cos \omega_2 t) B u \\ \vdots \\ 2 e^{\zeta_s t} e^{-\hat{N}_s t} (\hat{P}_s^R \cos \omega_s t + \hat{P}_s^I \sin \omega_s t) B u \\ 2 e^{\zeta_s t} e^{-\hat{N}_s t} (\hat{P}_s^R \sin \omega_s t - \hat{P}_s^I \cos \omega_s t) B u \end{bmatrix} \quad (2.4-25)$$

Or,

$$\delta = C \int_0^T \alpha(t) dt \quad (2.4-25)$$

$j = 1, 2, 3, \dots, s, \quad \lambda_j = -\zeta_j + i\omega_j \in \mathcal{C}^*$

in which we define

$$C = [c_1 \ c_2 \ \dots \ c_n] = [c_I \ \hat{c}_I]$$

$$\alpha(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} c_{II} \\ \hat{c}_{II} \end{bmatrix} \quad (2.4-27)$$

Thus, the displacement  $\delta$  can always be expressed in spatial form (Def. (2.3)) in real space  $R^n$  for the system (2.2-2) as shown in (2.4-26). At this point it is helpful to summarize the important steps involved in obtaining this expression.

### 2.4.3 Summary

The procedure to obtain the spatial form (Def. 2.3) in real space  $R^n$  for the state space displacement  $\delta$  for a linear time invariant system (2.2-2) is outlined in the following steps:

1. The equation to work with is

$$\delta = \int_0^T e^{-At} B u(t) dt$$

where  $A, B, u \in \dot{x} = Ax + Bu$ .

2. Obtain the eigenvalues of  $A$  and choose real eigenvectors (and generalized eigenvectors) for real eigenvalues and complex conjugate eigenvectors (and generalized eigenvectors) for complex conjugate eigenvalues, and construct the transformation matrix  $P$  so that  $P^{-1}AP = J$ . Also, obtain  $P^{-1}$ . The equation in step 1 can now be written as



$$\delta = \int_0^T P e^{-Jt} P^{-1} B u(t) dt$$

3. Partition  $P$  column wise and  $P^{-1}$  row wise so that each partition has  $\nu_k$  columns and rows, respectively, and corresponds to the diagonal block in  $e^{-Jt}$ . In partitioned form:

$$\delta = \int_0^T [\tilde{P}_1 \tilde{P}_2 \dots \tilde{P}_r] \begin{bmatrix} e^{-J_1 t} \hat{P}_1 B u(t) \\ e^{-J_2 t} \hat{P}_2 B u(t) \\ \vdots \\ e^{-J_r t} \hat{P}_r B u(t) \end{bmatrix} dt$$

If all eigenvalues are real then this is the spatial form for  $\delta$  in  $R^n$  given in (2.4-25), (2.4-26). On the other hand, if there are complex eigenvalues present then proceed to step 4.

4. Consider the pair of partitions in either matrix in the equation which corresponds to a pair of complex conjugate eigenvalues. Replace each one of this pair by a partition from the same location in  $\hat{C}_I$  or  $\hat{C}_I$  depending on the matrix considered. Repeat for the other matrix. Now,  $\delta$  is reduced to the spatial form in  $R^n$  given in (2.4-25), (2.4-26).

## 2.5 Test for Complete Controllability of the System

In article 2.4 we were concerned with obtaining the spatial form of  $\delta$  in real domain  $\mathbb{R}^n$ . In this article we will examine the spatial form of  $\delta$  and relate it to the controllability of the system (2.2-2).

### 2.5.1 Characteristics of the Spatial Form from the Point of View of Controllability

Let us consider the spatial form of  $\delta$  as defined in Def.(2.3). The time invariant columns  $c_j$  of  $C$ ,  $j=1,2,\dots,l$ , span a space whose dimension is less than or equal to  $l$  where  $l \leq n$ . If  $l < n$  then it is clear that the  $c_j$  vectors cannot span an  $n$  dimensional space. So any choice of the control vector  $u(t)$  for any arbitrary but finite time  $T$  can influence only those components of the initial state (or disturbed state) vector which lie in the subspace spanned by the  $c_j$  columns. As such we have no control over those components of the state vector which lie outside this subspace of the  $n$  dimensional space. Hence, an initial state (or disturbed state) which is such that it has one or more of these uncontrollable components cannot be brought to the origin in finite time. Therefore, the system is not completely controllable. This leads us to conclude that to avoid the certainty of uncontrollability  $l$  must not be less than  $n$ , or  $l = n$  and  $c_j$ ,  $j=1,2,\dots,n$  are all linearly independent (abbreviated L.I.). Now, by demanding that  $C$  be a full set of basis vectors for an  $n$  dimensional space we have created access to all  $n$  directions in the space. Is the

system now completely controllable? Since an initial state can have any arbitrary component in each of the  $n$  directions it is apparent that the components in the  $n$  directions must be affected independently over a finite time  $T$  if the system were to be completely controllable. The integral  $\int_0^T \alpha_j(t) dt$  influences the component in the  $c_j$  direction,  $j=1,2,\dots,n$ . Hence, the scalar functions  $\alpha_j(t)$  must be linearly independent over the period  $T$  for complete controllability of the system. Otherwise the system is uncontrollable. Thus we can conclude:

1. It is necessary that  $C$  be nonsingular so that the system may be controllable.
2. It is necessary that  $\alpha_j(t)$  be linearly independent over the period  $T$  so that the system may be completely controllable.
3. The two conditions (1) and (2) together are sufficient for complete controllability of the system.

Note that there is no single sufficient condition here. That is because we are discussing a hypothetical case in which  $C$  may be singular. In the next section we will show that the matrix  $C$  in our spatial form is always nonsingular thereby making the second condition sufficient for complete controllability.

We will illustrate the second condition for complete controllability by a couple of examples.

- a. Consider a system with scalar control  $u(t)$  with all eigenvalues real and  $n$  independent eigenvectors (no generalized vectors), so that

$c_j = p_j$ ,  $\alpha_j = e^{-\lambda_j t} \hat{p}_j^T b u$ , where  $p_j$ ,  $\hat{p}_j^T$ , etc. have the usual meaning and  $b=B$ . Now, if there are two identical eigenvalues,  $\lambda_j = \lambda_{j+1}$ , and the rest distinct, then

$$\begin{aligned} c_j \int_0^T \alpha_j(t) dt + c_{j+1} \int_0^T \alpha_{j+1}(t) dt \\ = p_j \hat{p}_j^T b \int_0^T u e^{-\lambda_j t} dt + p_{j+1} \hat{p}_{j+1}^T b \int_0^T u e^{-\lambda_{j+1} t} dt \\ = \left[ (\hat{p}_j^T b) p_j + (\hat{p}_{j+1}^T b) p_{j+1} \right] \int_0^T u e^{-\lambda_j t} dt \\ = \hat{c} \int_0^T u e^{-\lambda_j t} dt \end{aligned}$$

Thus, two directions  $c_j$ ,  $c_{j+1}$  collapse into one new direction  $\hat{c}$  and the spatial form is otherwise unaffected. This indicates loss of a basis vector and therefore leads to uncontrollability. Here, the fact that  $\lambda_j = \lambda_{j+1}$  made  $\alpha_j(t)$  and  $\alpha_{j+1}(t)$  linearly dependent (see Gramian test later).

b. In example (a) consider an  $m \times 1$  multiple control  $u(t)$ . Then

$$\begin{aligned} c_j \int_0^T \alpha_j(t) dt + c_{j+1} \int_0^T \alpha_{j+1}(t) dt \\ = \left[ p_j (\hat{p}_j^T B) + p_{j+1} (\hat{p}_{j+1}^T B) \right] \int_0^T u e^{-\lambda_j t} dt \end{aligned}$$

The bracket consists of  $m$  vectors which are linear combinations of  $c_j$  and  $c_{j+1}$ . If there are less than two L.I. vectors in this  $m$ -set then

it indicates loss of a basis vector, i.e., the system is uncontrollable. Otherwise it is controllable. The independence of  $\alpha_j(t)$  is related to the number of L.I. vectors in the  $m$ -set.

Another observation here is, if there are  $l$  repeated eigenvalues then we could combine  $l$  vectors of  $C$  in the same manner as in examples (a) and (b). In this case, in example (a) the result is always a single vector  $\hat{c}$  (indicating collapse of  $l-1$  vectors) with the conclusion that it is an uncontrollable system. In example (b) as long as  $l \leq m$  there is a possibility that the  $m$ -set will not degenerate below  $l$  L.I. vectors. If  $l > m$  then  $l-m$  vectors have already collapsed with the certainty of uncontrollability.

In this discussion we have highlighted the characteristics of the spatial form  $\delta$  from the controllability point of view. In the next couple of sections we will derive rigorously the necessary and sufficient conditions for complete controllability for linear time invariant systems (2.2-2).

### 2.5.2 Nonsingularity of the C Matrix

In section 2.5.1 we examined the spatial form of  $\delta$  in its most general form (Def.2.3) for characteristics of controllability. We concluded that two necessary conditions have to be satisfied simultaneously to guarantee complete controllability of the system. In the following theorem it is shown that for the spatial form of  $\delta$  we derived in (2.4-23) (or (2.4-25)) the C matrix is always nonsingular.

Theorem 2.2: In the spatial form of the displacement  $\delta$  in real space, derived in theorem 2.1 the C matrix is always nonsingular.

Proof: If all the eigenvalues of the system are real then the C matrix is the transformation matrix P and hence is always nonsingular. But, when there are complex eigenvalues present the C matrix is slightly modified, i.e., a pair of complex conjugate eigenvectors (or generalized eigenvectors) are replaced by the real part and the imaginary part of the vector in the pair which corresponds to  $\lambda_j = -\zeta_j + i \omega_j$ . All we need here is to prove that all the n vectors of C are linearly independent (their positions in C do not matter).

Suppose that there are q pairs of complex conjugate eigenvalues (n-2q real eigenvalues). Let us rearrange the columns of P so that

$$P_{\text{new}} = [P_1 \ P_2 \ P_3] \quad (2.5-1)$$

where  $P_1$  (nx(n-2q)) is a set of all real eigenvectors (and generalized eigenvectors) and  $P_2$  (nxq),  $P_3$  (nxq) are sets such that for every vector in  $P_2$  its conjugate is in  $P_3$  (conjugate property of eigenvectors). Thus,  $P_3 = \bar{P}_2$ . The inverse of  $P_{\text{new}}$  can be obtained by corresponding rearrangement of the rows of  $P^{-1}$ . The matrix  $C_{\text{new}}$  (rearranged form of C) is obtained simply by replacing  $P_2$  and  $P_3$  in  $P_{\text{new}}$  by  $P_2^R$  and  $P_2^I$ , respectively, where  $P_2 = P_2^R + i P_2^I$ . We now have the following matrices:

$$P_{\text{new}} = [P_1 \ P_2 \ \bar{P}_2] \\ C_{\text{new}} = [P_1 \ P_2^R \ P_2^I] ; \text{ and } P_{\text{new}}^{-1} \quad (2.5-2)$$

But  $P_{new}^{-1} P_{new} = E$  ( $n \times n$ ), the unit matrix, and hence substituting for  $P_{new}$  from (2.5-2)

$$\begin{aligned} P_{new}^{-1} P_{new} &= P_{new}^{-1} [P_1 \ P_2 \ \bar{P}_2] = [P_{new}^{-1} P_1 \ P_{new}^{-1} P_2 \ P_{new}^{-1} \bar{P}_2] \\ &= \text{diag} [E_1 \ E_2 \ E_3] \end{aligned} \quad (2.5-3)$$

where  $E_1, E_2, E_3$  are unit matrices of order  $(n-2q), q, q$ , respectively. Equating the left side and the right side partition wise in the above equation, we have the following result:

$$P_{new}^{-1} P_1 = \begin{bmatrix} E_1 \\ 0 \\ 0 \end{bmatrix} ; \quad P_{new}^{-1} P_2 = \begin{bmatrix} 0 \\ E_2 \\ 0 \end{bmatrix} ; \quad P_{new}^{-1} \bar{P}_2 = \begin{bmatrix} 0 \\ 0 \\ E_2 \end{bmatrix} \quad (2.5-4)$$

Now, substituting for  $C_{new}$  from (2.5-2),

$$P_{new}^{-1} C_{new} = P_{new}^{-1} [P_1 \ P_2^R \ P_2^I] = [P_{new}^{-1} P_1 \ P_{new}^{-1} P_2^R \ P_{new}^{-1} P_2^I] \quad (2.5-5)$$

We can obtain  $P_{new}^{-1} P_2^R, P_{new}^{-1} P_2^I$  as follows:

$$P_{new}^{-1} P_2 + P_{new}^{-1} \bar{P}_2 = P_{new}^{-1} (P_2 + \bar{P}_2) = P_{new}^{-1} (2 P_2^R)$$

$$P_{new}^{-1} P_2^R = \frac{1}{2} (P_{new}^{-1} P_2 + P_{new}^{-1} \bar{P}_2)$$

and using (2.5-4)

$$P_{\text{new}}^{-1} P_2^R = \frac{1}{2} \begin{bmatrix} 0 \\ E_2 \\ E_2 \end{bmatrix} \quad (2.5-6)$$

Similarly,

$$P_{\text{new}}^{-1} P_2^I = -\frac{i}{2} (P_{\text{new}}^{-1} P_2 - P_{\text{new}}^{-1} \bar{P}_2)$$

or, using (2.5-4)

$$P_{\text{new}}^{-1} P_2^I = -\frac{1}{2} i \begin{bmatrix} 0 \\ E_2 \\ -E_2 \end{bmatrix} \quad (2.5-7)$$

Substituting (2.5-4), (2.5-5) and (2.5-7) in (2.5-5),

$$P_{\text{new}}^{-1} C_{\text{new}} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & \frac{1}{2} E_2 & -\frac{i}{2} E_2 \\ 0 & \frac{1}{2} E_2 & \frac{i}{2} E_2 \end{bmatrix} \quad (2.5-8)$$

The rank of the matrix on the right side in (2.5-8) above is  $n$ , and hence  $P_{\text{new}}^{-1} C_{\text{new}}$  is nonsingular. Therefore,  $C_{\text{new}}$  is nonsingular. This proves the theorem.

Thus, the first necessary condition that  $C$  must be nonsingular to avoid certainty of uncontrollability (see section 2.5.1) is always satisfied for the spatial form of  $\delta$  in real space  $R^n$ . This leads us to conclude that the second necessary condition, i.e., the requirement



4-

of the linear independence of the scalar time dependent functions  $\alpha_j(t)$  of  $\alpha(t)$  over the period  $T$  is also a sufficient condition for complete controllability of the system. Hence, it is only necessary to test for the linear independence of the  $\alpha_j(t)$ . But as it stands now it is not easy to conduct this test, and it is desirable to derive some conditions which will be necessary and sufficient to guarantee the independence of the  $\alpha_j(t)$ . The next section is devoted to this objective.

### 2.5.3 Necessary and Sufficient Conditions for Complete Controllability for Linear Time Invariant Systems

In this section we will derive a set of necessary and sufficient conditions which will guarantee the independence of  $\alpha_j(t)$  over the period  $T$ , thereby guaranteeing complete controllability of a linear time invariant system.

For the spatial form of  $\delta$  in real space (see (2.4-25)), from (2.4-25),

$$\alpha(t) = \begin{bmatrix} \mathcal{C}_{II} \\ \hat{\mathcal{C}}_{II} \end{bmatrix}$$

where  $\mathcal{C}_{II}$  is the set of vector partitions  $\{e^{-\mathcal{J}_j t} \hat{p}_j B u\}$  for all  $\lambda_j \in \mathcal{C}$ , and  $\hat{\mathcal{C}}_{II}$  is a similar set of pairs of vector partitions

$$\left\{ \begin{array}{l} 2 e^{\hat{\lambda}_j t} e^{-\hat{N}_j t} (\hat{p}_j^R \cos \omega_j t + \hat{p}_j^I \sin \omega_j t) B u \\ 2 e^{\hat{\lambda}_j t} e^{-\hat{N}_j t} (\hat{p}_j^R \sin \omega_j t - \hat{p}_j^I \cos \omega_j t) B u \end{array} \right\}$$

46

for all  $\lambda_j, \bar{\lambda}_j \in \mathcal{C}^*$  in which  $\lambda_j = -\zeta_j + i\omega_j$  (see (2.4-24)). Each vector partition consists of  $\nu_j$  elements (scalar time dependent functions) where  $\nu_j$  is the order of the Jordan block  $J_j$ . Within each vector partition all the  $\nu_j$  elements are linearly independent over the period  $T$  provided there is no zero element present. (These elements are members of a Jordan chain; each element differs from the other in the highest power of  $t$ , because of the polynomial in  $t$  in the exponential matrix  $e^{-\hat{N}_j t}$  (see (2.4-3)).) Furthermore, between any two vector partitions in  $\mathcal{C}_{II}$  if the exponential factor  $e^{-\lambda_j t}$  of one differs from that of the other then these two sets are linearly independent of each other over the period  $T$ . Similarly, between any two vector partitions in  $\hat{\mathcal{C}}_{II}$  if either  $\zeta_j$  or  $\omega_j$  of one differ from that of the other, then due to the exponential factor  $e^{\zeta_j t}$  or the factors  $\cos \omega_j t$ ,  $\sin \omega_j t$ , these two sets are L.I. over the time  $T$ . Also any vector partition of  $\mathcal{C}_{II}$  is L.I. of any vector partition of  $\hat{\mathcal{C}}_{II}$  over the period  $T$  due to the presence of the circular functions of  $\omega_j$  in  $\hat{\mathcal{C}}_{II}$ . Thus, the test can be narrowed down to checking the vector partitions in  $\mathcal{C}_{II}$  and  $\hat{\mathcal{C}}_{II}$  separately, and 1) those partitions in  $\mathcal{C}_{II}$  for which  $\lambda_j$  are identical, and 2) those partitions in  $\hat{\mathcal{C}}_{II}$  for which both  $\zeta_j$ ,  $\omega_j$  are identical. A further simplification is obtained by using the Jordan chain property for the elements of a vector partition, i.e., if the  $\nu_j$ th element of one vector partition is independent of the  $\nu_j$ th element of another vector partition then the two sets are independent of each other. Because the elements of a vector partition are a chain built over this  $\nu_j$ th element. (An analogous reference is the modulation vector of Def.2.4. Actually the

$\nu_j$ th element is made up of the foundation vector.) For convenience we will call these  $\nu_j$ th elements in each vector partition as "foundation" elements in  $\alpha(t)$ . Thus, finally the test for linear independence need be conducted only for the foundation elements of:

- 1) the partitions in  $\mathcal{C}_{\mathbb{R}}$  for which  $\lambda_j$  are identical, for all  $\lambda_j \in \mathcal{C}$ ; and
- 2) the partitions in  $\mathcal{C}_{\mathbb{C}}$  for which both  $\zeta_j, \omega_j$  are identical, for all  $\lambda_j \in \mathcal{C}^*$ .

Let  $\hat{\lambda}_j$  (in  $\mathcal{C}$  or  $\mathcal{C}^*$ ) denote distinct eigenvalues of the system, and let  $\mu_j$  be the number of Jordan blocks  $J_k$  (each  $J_k$  associated with a  $\lambda_k$ ,  $k=1,2,\dots,r$ ,  $r \leq n$ ), for which  $\lambda_k$  are identical and equal to  $\hat{\lambda}_j$ . Let  $S_j$  be sets of foundation elements of  $\alpha(t)$ : 1) Corresponding to  $\hat{\lambda}_j$  for  $\hat{\lambda}_j \in \mathcal{C}$ , and 2) Corresponding to  $\hat{\lambda}_j, \bar{\hat{\lambda}}_j$  for  $\hat{\lambda}_j \in \mathcal{C}^*$ . Since each Jordan block gives rise to one foundation element each  $S_j$  contains  $\mu_j$  elements if  $\hat{\lambda}_j$  is real, and  $2\mu_j$  elements if  $\hat{\lambda}_j$  is complex. For the complex case,  $S_j$  is a set of all elements for which both  $\zeta_k$  and  $\omega_k$  ( $\lambda_k = -\zeta_k + i\omega_k$ ) are identical. Let  $\gamma_j$  denote: 1) sets of foundation vectors (see Def.2.4)  $\hat{p}_{Lk}$  corresponding to all the Jordan blocks  $J_k$  associated with  $\hat{\lambda}_j$  for  $\hat{\lambda}_j \in \mathcal{C}$ , or 2) Sets of the real and imaginary parts  $\hat{p}_{Lk}^R, \hat{p}_{Lk}^I$  of the foundation vectors  $\hat{p}_{Lk}$  corresponding to all the Jordan blocks  $J_k$  associated with either one of a complex pair of  $\hat{\lambda}_j, \bar{\hat{\lambda}}_j \in \mathcal{C}^*$ . The sets  $\gamma_j$  contain  $\mu_j$  vectors (columns) for  $\hat{\lambda}_j \in \mathcal{C}$  and  $2\mu_j$  vectors for  $\hat{\lambda}_j \in \mathcal{C}^*$ . Or,

$$\gamma_j = \begin{cases} [\gamma_1 \ \gamma_2 \ \dots \ \gamma_{\mu_j}] , & \hat{\lambda}_j \in \mathcal{C} \\ \begin{bmatrix} \gamma_1^R & \gamma_1^I & \gamma_2^R & \gamma_2^I & \dots & \gamma_{\mu_j}^R & \gamma_{\mu_j}^I \end{bmatrix} , & \hat{\lambda}_j \in \mathcal{C}^* \end{cases} \quad (2.5-9)$$

where  $y_i$  or  $y_i^R$ ,  $y_i^I$  are used in place of  $\hat{p}_{Lk}$ , etc. to simplify notation. Now, using (2.4-2), (2.4-3) and (2.4-7) in (2.4-24) for the foundation elements of  $\alpha(t)$  (in  $C_I$ ,  $C_I$ ) the sets  $S_j$  can be written as follows:

$$S_j = [e^{-\hat{\lambda}_j t} y_1^T B u, e^{-\hat{\lambda}_j t} y_2^T B u, \dots, e^{-\hat{\lambda}_j t} y_{n_j}^T B u]$$

for  $\hat{\lambda}_j \in C$

$$S_j = \begin{bmatrix} 2 e^{\hat{\lambda}_j t} \{ (y_1^R)^T B u \cos \hat{\omega}_j t + (y_1^I)^T B u \sin \hat{\omega}_j t \}, \\ 2 e^{\hat{\lambda}_j t} \{ (y_1^R)^T B u \sin \hat{\omega}_j t - (y_1^I)^T B u \cos \hat{\omega}_j t \}, \\ 2 e^{\hat{\lambda}_j t} \{ (y_2^R)^T B u \cos \hat{\omega}_j t + (y_2^I)^T B u \sin \hat{\omega}_j t \}, \\ 2 e^{\hat{\lambda}_j t} \{ (y_2^R)^T B u \sin \hat{\omega}_j t - (y_2^I)^T B u \cos \hat{\omega}_j t \}, \end{bmatrix}$$

$$\begin{bmatrix} 2 e^{\hat{\lambda}_j t} \{ (y_{n_j}^R)^T B u \cos \hat{\omega}_j t + (y_{n_j}^I)^T B u \sin \hat{\omega}_j t \} \\ 2 e^{\hat{\lambda}_j t} \{ (y_{n_j}^R)^T B u \sin \hat{\omega}_j t - (y_{n_j}^I)^T B u \cos \hat{\omega}_j t \} \end{bmatrix}$$

for  $\hat{\lambda}_j \in C^*$

(2.5-10)

Note that the elements of  $S_j$  are scalars. For  $\hat{\lambda}_j \in C$ ,  $S_j$  can be expressed in the matrix form

ORIGINAL FILED  
OF PROOF

$$S_j = e^{-\hat{\lambda}_j t} u^T B^T \gamma_j, \quad \hat{\lambda}_j \in \mathcal{C} \quad (2.5-11)$$

where  $\gamma_j^T B u = u^T B^T \gamma_j$ , etc., and  $\gamma_j$  is as given in (2.5-9) for  $\hat{\lambda}_j \in \mathcal{C}$ . To express  $S_j$  for  $\hat{\lambda}_j \in \mathcal{C}^*$  in a similar form, let  $u_a = u \cos \hat{\omega}_j t$  and  $u_b = u \sin \hat{\omega}_j t$ . Since each element of  $S_j$  is a sum of two components,  $S_j$  can be written as a sum of two sets (matrices) (with a slight rearrangement† for convenient algebraic signs) as shown below:

$$S_j = 2 e^{\hat{\lambda}_j t} u_a^T B^T [-y_1^I, y_1^R, -y_2^I, y_2^R, \dots, -y_{\mu_j}^I, y_{\mu_j}^R] \\ + 2 e^{\hat{\lambda}_j t} u_b^T B^T [y_1^R, y_1^I, y_2^R, y_2^I, \dots, y_{\mu_j}^R, y_{\mu_j}^I] \quad (2.5-12)$$

Further, if we define

$$K = \text{diag} [K', K', K', \dots, \mu_j \text{ blocks } \dots, K'] \\ K' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (2.5-13)$$

then (2.5-12) can be expressed as

$$S_j = 2 e^{\hat{\lambda}_j t} [u_a^T \ u_b^T] \begin{bmatrix} B^T \gamma_j K \\ B^T \gamma_j \end{bmatrix}, \quad \hat{\lambda}_j \in \mathcal{C}^* \quad (2.5-14)$$

†The order of elements in a set does not affect their relationship with respect to independence or dependence.

where  $Y_j$  is as defined in (2.5-9) for  $\hat{\lambda}_j \in \mathcal{C}^*$ .

The time dependent scalar functions of  $S_j$  in (2.5-10) are said to be linearly independent over the time interval  $[0, T]$  if the Gramian matrix associated with  $S_j$  is nonsingular. More details on Gramian matrices can be found in [2]. The Gramians  $G_j$  associated with our sets  $S_j$  are given by

$$G_j = \int_0^T S_j^T S_j \quad (2.5-15)$$

Substituting for  $S_j$  from (2.5-11) and (2.5-14) and noting that  $B, Y_j, K$  are time invariant, (2.5-15) takes the form

$$G_j = \begin{cases} (B^T Y_j)^T M (B^T Y_j) , & \hat{\lambda}_j \in \mathcal{C} \\ [(B^T Y_j, K)^T (B^T Y_j)^T] M \begin{bmatrix} B^T Y_j, K \\ B^T Y_j \end{bmatrix} , & \hat{\lambda}_j \in \mathcal{C}^* \end{cases} \quad (2.5-16)$$

where

$$M = \int_0^T e^{-2\hat{\lambda}_j t} u u^T dt , \quad \hat{\lambda}_j \in \mathcal{C}$$

$$M = \int_0^T \frac{1}{4} e^{2\hat{\lambda}_j t} \begin{bmatrix} u_a u_a^T & u_a u_b^T \\ u_b u_a^T & u_b u_b^T \end{bmatrix} dt , \quad \hat{\lambda}_j \in \mathcal{C}^*$$

(2.5-17)

The matrix  $M$  is of order  $m$  for  $\hat{\lambda}_j \in \mathcal{C}$  , and of order  $2m$  for  $\hat{\lambda}_j \in \mathcal{C}^*$ .

The Gramian  $G_j$  is of order  $\mu_j$  for  $\hat{\lambda}_j \in \mathcal{C}$  and of order  $2\mu_j$  for  $\hat{\lambda}_j \in \mathcal{C}^*$ . Hence, for linear independence of all  $\alpha_i(t)$  in  $\alpha(t)$ ,  $i=1,2,\dots,n$ , all the Gramians  $G_j$  must be nonsingular, i.e., rank  $G_j$  must be equal to  $\mu_j$  for  $\hat{\lambda}_j \in \mathcal{C}$ , and rank  $G_j$  must be equal to  $2\mu_j$  for  $\hat{\lambda}_j \in \mathcal{C}^*$ . From (2.5-15) and (2.5-17) we can write

$$\text{rank } G_j \leq \text{rank } M$$

$$\text{rank } G_j \leq \text{rank } B^T Y_j, \quad \hat{\lambda}_j \in \mathcal{C}$$

and

$$\text{rank } G_j \leq \text{rank } M$$

$$\text{rank } G_j \leq \text{rank} \begin{bmatrix} B^T Y_j k \\ B^T Y_j \end{bmatrix} \quad (2.5-18)$$

$$\hat{\lambda}_j \in \mathcal{C}^*$$

Now, we are ready to formulate the necessary and sufficient conditions for complete controllability of the system. We prove this in the following theorem.

**Theorem 2.3:** In a linear time invariant system

$$\dot{x} = Ax + Bu \quad (2.5-19)$$

where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , let  $J$  be the Jordan canonical form of  $A$  and  $P$  be the transformation matrix such that  $P^{-1}AP = J$ . Let  $J_k$  be the Jordan blocks of order  $\nu_k$ . Associated with each  $J_k$  is an eigenvalue  $\lambda_k$ ,  $k=1,2,\dots,r$ ,  $r \leq n$ . Let  $\hat{\lambda}_j$ ,  $j=1,2,\dots,l$  denote distinct

eigenvalues so that  $l \leq r$ . Let  $\mu_j$  be the number of Jordan blocks  $J_k$  for which  $\lambda_k$  are identical and equal to  $\hat{\lambda}_j$ . Let  $\gamma_j$  be: 1) Sets of foundation vectors [see Def.(2.4)]  $\hat{p}_{Lk}$  corresponding to the Jordan blocks  $J_k$  associated with  $\hat{\lambda}_j$ ,  $\hat{\lambda}_j \in \mathcal{C}$ , or 2) Sets of the real and imaginary parts of  $\hat{p}_{Lk}$  (i.e.,  $\hat{p}_{Lk}^R$  and  $\hat{p}_{Lk}^I$ ) corresponding to the Jordan blocks  $J_k$  associated with  $\hat{\lambda}_j$  or  $\bar{\hat{\lambda}}_j$ ,  $\hat{\lambda}_j, \bar{\hat{\lambda}}_j \in \mathcal{C}^*$ . Then:

1. The necessary conditions to be satisfied so that the system may be completely controllable is

$$m \geq \mu_j \quad \forall \text{ distinct } \hat{\lambda}_j, j=1, 2, \dots, l \quad (2.5-20)$$

where  $m$  is the maximum available number of independently variable scalar controls  $u_i(t)$  belonging to  $u(t)$  (i.e., dimension of  $u(t)$ ).

Note: For  $\hat{\lambda}_j \in \mathcal{C}^*$  only one of a pair of complex conjugate  $\hat{\lambda}_j$ ,  $\bar{\hat{\lambda}}_j$ , need be considered.

2. The necessary and sufficient condition to be satisfied so that the system will be completely controllable is

$$\begin{aligned} \text{rank } B^T \gamma_j &= \text{rank } \gamma_j = \mu_j \quad \forall \text{ distinct } \hat{\lambda}_j \in \mathcal{C} \\ \text{rank } \begin{bmatrix} B^T \gamma_{jk} \\ B^T \gamma_j \end{bmatrix} &= \text{rank } \gamma_j = 2\mu_j \quad \forall \text{ distinct } \hat{\lambda}_j \in \mathcal{C}^* \end{aligned} \quad (2.5-21)$$

Note: For  $\hat{\lambda}_j \in \mathcal{C}$   $\text{rank } \gamma_j = \mu_j$  and for  $\hat{\lambda}_j \in \mathcal{C}^*$   $\text{rank } \gamma_j = 2\mu_j$ . Again for  $\hat{\lambda}_j \in \mathcal{C}^*$  only one of a pair of complex



conjugate  $\hat{\lambda}_j, \bar{\hat{\lambda}}_j$  need be considered.

Proof: 1. Necessary condition - To prove the necessary condition (2.5-20) we will prove that if this is not satisfied then it is a sufficient condition for uncontrollability of the system. Hence, assume

$$m < \mu_j \text{ for some } \hat{\lambda}_j, j = 1, 2, \dots, l \quad (2.5-22)$$

Consider the two cases:

a)  $\hat{\lambda}_j \in \mathcal{C}$

The matrix  $B$  is  $n \times m$  and  $Y_j$  is  $n \times \mu_j$ , so that  $B^T Y_j$  is  $m \times \mu_j$ , and  $\text{rank } B^T Y_j \leq m$ . But from (2.5-18)  $\text{rank } G_j \leq \text{rank } B^T Y_j$ . Hence, if  $m < \mu_j$ ,

$$\text{rank } G_j \leq \text{rank } B^T Y_j \leq m < \mu_j$$

$$\text{rank } G_j < \mu_j \quad (2.5-23)$$

Thus, the Gramian is singular for this  $\hat{\lambda}_j \in \mathcal{C}$  which indicates uncontrollability of the system (due to the existence of a dependent set  $S_j$  in  $\alpha(t)$ ).

b)  $\hat{\lambda}_j \in \mathcal{C}^*$

The matrix  $Y_j$  is of dimension  $n \times 2\mu_j$  so that  $B^T Y_j$  is  $m \times 2\mu_j$ . From (2.5-13)  $K$  is a square matrix of order  $2\mu_j$ , so  $B^T Y_j K$  is also of dimension  $m \times 2\mu_j$ . If we call

$$\gamma_j = \begin{bmatrix} B^T Y_j K \\ B^T Y_j \end{bmatrix} \quad (2.5-24)$$

then  $\gamma_j$  is of dimension  $2m \times 2\mu_j$ . And  $\text{rank } \gamma_j \leq 2m$ . But from (2.5-13)

$\text{rank } G_j \leq \text{rank } \gamma_j$ . Hence, if  $m < \mu_j$ , i.e.,  $2m < 2\mu_j$ ,

$$\text{rank } G_j \leq \text{rank } \gamma_j \leq 2m < 2\mu_j$$

or

$$\text{rank } G_j < 2\mu_j \quad (2.5-25)$$

i.e., the Gramian is singular for this  $\hat{\lambda}_j \in \mathbb{C}^*$ , and this is sufficient condition for uncontrollability of the system.

This shows that the condition  $m < \mu_j$  for some  $\hat{\lambda}_j$  (real or complex),  $j=1,2,\dots,l$ , is sufficient for uncontrollability of the system. Hence, we conclude that the condition

$$m \geq \mu_j \quad \forall \text{ distinct } \hat{\lambda}_j, j=1,2,\dots,l$$

is necessary so that the system may be completely controllable. This proves the first part of the theorem.

## 2. Necessary and Sufficient Condition

a) To prove the necessary part consider the ranks of  $B^T \gamma_j$  and  $\gamma_j$  whose dimensions are  $m \times \mu_j$  and  $2m \times 2\mu_j$ , respectively. If  $\text{rank } B^T \gamma_j \neq \mu_j$  then  $\text{rank } B^T \gamma_j < \mu_j$ . Similarly, if  $\text{rank } \gamma_j \neq 2\mu_j$ , then  $\text{rank } \gamma_j < 2\mu_j$ . Hence, from (2.5-13),

$$\text{rank } G_j \leq \text{rank } B^T \gamma_j < \mu_j, \quad \hat{\lambda}_j \in \mathbb{C}$$

$$\text{rank } G_j \leq \text{rank } \gamma_j < 2\mu_j, \quad \hat{\lambda}_j \in \mathbb{C}^*$$

Or,

$$\begin{aligned} \text{rank } G_j &< \mu_j, \quad \hat{\lambda}_j \in C \\ \text{rank } G_j &< 2\mu_j, \quad \hat{\lambda}_j \in C^* \end{aligned} \quad (2.5-26)$$

and from the first part we know this leads to uncontrollability of the system if (2.5-26) is true for any  $\hat{\lambda}_j$ .

Hence, the necessary conditions for controllability are:

$$\begin{aligned} \text{rank } B^T \gamma_j &= \mu_j \quad \text{for } \hat{\lambda}_j \in C \\ \text{rank } \gamma_j &= 2\mu_j \quad \text{for } \hat{\lambda}_j \in C^* \end{aligned} \quad (2.5-27)$$

which imply

$$m \geq \mu_j \quad \text{for all } \hat{\lambda}_j \in C, C^* \quad (2.5-28)$$

Also, rank  $\gamma_j$  is  $\mu_j$  for  $\hat{\lambda}_j \in C$ , and  $2\mu_j$  for  $\hat{\lambda}_j \in C^*$ , because  $\gamma_j$  are sets of left eigenvectors or their derivatives (real and imaginary parts; for this case the proof of L.I. is similar to theorem (2.2)).

Therefore, (2.5-27) can also be expressed as:

$$\begin{aligned} \text{rank } B^T \gamma_j &= \text{rank } \gamma_j = \mu_j, \quad \hat{\lambda}_j \in C \\ \text{rank } \gamma_j &= \text{rank } \gamma_j = 2\mu_j, \quad \hat{\lambda}_j \in C^* \end{aligned} \quad (2.5-29)$$

b) To prove the sufficient condition consider the quadratic form in  $v$ ,  $v^T G_j v$  where: 1)  $v \in R^{\mu_j}$  for  $\hat{\lambda}_j \in C$ , or 2)  $v \in R^{2\mu_j}$  for  $\hat{\lambda}_j \in C^*$ , and  $G_j$  is as given in (2.5-15). We can then write

$$v^T G_j v = \begin{cases} (B^T \gamma_j v)^T M (B^T \gamma_j v) & , \hat{\lambda}_j \in \mathcal{C} \\ (\gamma_j v)^T M (\gamma_j v) & , \hat{\lambda}_j \in \mathcal{C}^* \end{cases} \quad (2.5-30)$$

where  $M$  is as defined in (2.5-17) and  $\gamma_j$  as in (2.5-24). Also, let

$$z = \begin{cases} B^T \gamma_j v & , \hat{\lambda}_j \in \mathcal{C} \\ \gamma_j v & , \hat{\lambda}_j \in \mathcal{C}^* \end{cases} \quad (2.5-31)$$

where  $z \in \mathbb{R}^m$  for  $\hat{\lambda}_j \in \mathcal{C}$ , and  $z \in \mathbb{R}^{2m}$  for  $\hat{\lambda}_j \in \mathcal{C}^*$ . Then

$$v^T G_j v = z^T M z \quad (2.5-32)$$

It is now useful to recall the concept of controllability itself. When we ask if a system is completely controllable we imply all the freedom, if necessary, in the choice of scalar controls  $u_i(t)$  for a given number of them ( $i=1,2,\dots,m$ ), i.e., the scalar controls  $u_i(t)$ ,  $i=1,2,\dots,m$  are independently variable. Given this freedom we ask if the system can ever be controlled. Hence, we are at liberty to use all the freedom available for a given set of scalar controls.

Now,

$$M = \int_0^T (e^{-\hat{\lambda}_j t} u) (e^{-\hat{\lambda}_j t} u)^T dt, \quad \hat{\lambda}_j \in \mathcal{C}$$

$$M = \int_0^T \begin{bmatrix} (2e^{\hat{\lambda}_j t} \cos \hat{\omega}_j t) u \\ (2e^{\hat{\lambda}_j t} \sin \hat{\omega}_j t) u \end{bmatrix} \begin{bmatrix} (2e^{\hat{\lambda}_j t} \cos \hat{\omega}_j t) u \\ (2e^{\hat{\lambda}_j t} \sin \hat{\omega}_j t) u \end{bmatrix}^T dt \quad (2.5-33)$$

$$\hat{\lambda}_j \in \mathcal{C}^*$$

We see that  $M$  is a Gramian matrix for both cases associated with a set  $f(t)$  of scalars, i.e.,

$$f(t) = \begin{cases} e^{-\hat{\lambda}_j t} u^T, & \hat{\lambda}_j \in \mathcal{C} \\ \left[ (2e^{\hat{\lambda}_j t} \cos \hat{\omega}_j t) u^T, (2e^{\hat{\lambda}_j t} \sin \hat{\omega}_j t) u^T \right], & \hat{\lambda}_j \in \mathcal{C}^* \end{cases} \quad (2.5-34)$$

Since  $u_i(t)$  can be varied independently for all  $i, i=1, 2, \dots, m$  over the period  $T$ , the scalars of  $f(t)$  are also independently variable over  $T$ . (exponential factors do not affect the independence of  $u_i$ , and  $\cos \hat{\omega}_j t$  and  $\sin \hat{\omega}_j t$  being orthogonal functions preserve the independence of  $f_i$ ). As such the Gramians associated with sets  $f(t)$  can be made nonsingular. Hence, assume  $M$  can be made positive definite ( $M > 0$ ).

Or, looking at another way,

$$\begin{aligned} z^T M z &= z^T \left\{ \int_0^T f(t) f(t) dt \right\} z \\ &= \int_0^T z^T f^T(t) f(t) z dt \\ &= \int_0^T w^T(t) w(t) dt \end{aligned} \quad (2.5-35)$$

where  $z$  is a constant vector, and  $w(t) = f(t)z$  is a scalar. If  $z^T M z = 0$  for all time  $t \in [0, T]$  then from (2.5-35)  $w(t) = 0$  is the only possible solution, or  $f(t)z = 0$ . But  $f(t)$  is a set of scalars which can be varied independently over the period  $T$ .

So for this equation to be true for all time  $t \in [0, T]$ ,  $z$  must be zero. In other words, if  $z \neq 0$ , for an independent set  $f(t)$ ,

$$\int_0^T w(t)^2 dt$$

can always be made positive for a finite time  $T$ . i.e.,  $z^T M z > 0$ ,  $z \neq 0$ . Thus, a positive definite  $M$  can be assumed as within our capability for controllability purpose. Hence, assuming  $M > 0$ , we know that if  $z^T M z = 0$ , then  $z = 0$  is the only solution possible. Or, from (2.5-31),

$$\begin{aligned} z &= B^T \gamma; v = 0, \quad \hat{\lambda}_j \in C \\ z &= \gamma_j; v = 0, \quad \hat{\lambda}_j \in C^* \end{aligned} \quad (2.5-36)$$

Since  $v$  is  $M_j \times 1$  for  $\hat{\lambda}_j \in C$ , if  $\text{rank } B^T \gamma_j = M_j$  then  $v = 0$  is the unique solution to  $B^T \gamma_j; v = 0$ . Similarly,  $v$  is  $2M_j \times 1$  for  $\hat{\lambda}_j \in C^*$ , and if  $\text{rank } \gamma_j = 2M_j$  then  $v = 0$  is the only solution to  $\gamma_j; v = 0$ . But this implies  $v = 0$  is the only solution to  $v^T G_j; v = 0$  ( $z^T M z = 0$ ). This will require  $G_j$  to be nonsingular. Hence, the conditions (2.5-29) for all  $\hat{\lambda}_j \in C, C^*$  are sufficient to guarantee nonsingularity of all  $G_j$ . Combining with part (a) the necessary and sufficient conditions for complete controllability are, therefore, given by (2.5-29). This completes the proof.

In the above theorem we have derived the necessary and sufficient conditions for complete controllability of the system. Although the necessary condition (1) is implicit in the necessary and sufficient

conditions in (2), the former can serve as a quick and simple check to see if the system is definitely uncontrollable.

Even though the above conditions were arrived at by a rather lengthy process their application in testing for the complete controllability of the system is not very complicated. The first necessary condition is very simple (involving only counting of the Jordan blocks) once the Jordan canonical form is obtained. In the second condition the rank test involves only much smaller matrices ( $m \times A_j$ ,  $m \times 2A_j$ ), and  $m$  and  $A_j$  are generally much smaller than the dimension  $n$  of the system. This will reduce numerical errors in the computer methods. Moreover, the first test provides a minimum number for the scalar controls  $u_j$  that is absolutely necessary so that the system may be completely controllable.

In this section we have been concerned with the independence of the scalar functions of  $\alpha(t)$  in the equation for  $\phi$  in spatial form in real space. We derived the necessary and sufficient conditions which would guarantee this independence. In the following we summarize the practical steps involved to conduct the test for complete controllability of the system.

1. Obtain the Jordan canonical form  $J$  of the matrix  $A$ . Count the number of Jordan blocks associated with each distinct eigenvalue. If this number exceeds the dimension of the control vector  $u(t)$  for any distinct eigenvalue then the system is certainly uncontrollable. If it passes this test proceed to step 2.

2. Set up the  $Y_j$  matrices from the foundation vectors corresponding to the set of Jordan blocks associated with each distinct eigenvalue  $\hat{\lambda}_j$ . (For complex  $\hat{\lambda}_j$  the real and imaginary parts of the foundation vectors must be used.) Test the rank of  $B^T Y_j$  for all  $\hat{\lambda}_j \in \mathcal{C}$ , and test the rank of  $Y_j$  for all  $\hat{\lambda}_j \in \mathcal{C}^*$ . If their ranks are equal to the ranks of  $Y_j$  (for  $\mathcal{C}, \mathcal{C}^*$  as the case may be) for all distinct  $\hat{\lambda}_j$  then the system is completely controllable. If not it is uncontrollable.

Some implications of these conditions of controllability are worth noting. Consider a scalar control system ( $m = 1, u = u_j$ ). From the necessary conditions for controllability, we have

$$\mu_j \leq m = 1$$

or

$$\mu_j \leq 1 \quad (2.5-37)$$

This implies that no two Jordan blocks  $J_k$  of  $J$  should be associated with the same eigenvalue  $\hat{\lambda}_j$  (i.e.,  $\lambda_k, k=1,2,\dots,r$  must all be distinct), in order that the system may be completely controllable. Further results can be stated in the following theorem.

Theorem 2.4: If all the Jordan blocks  $J_k, k=1,2,\dots,r$ , of the system (2.5-19) are associated with distinct eigenvalues  $\hat{\lambda}_j$ , i.e.,  $j = k$ , then the system is completely controllable if none of the components of the vector  $\alpha(t)$  is zero, i.e.,

$$\alpha_i(t) \neq 0 \quad \text{for any } i, i = 1,2,\dots,n \quad (2.5-38)$$



Proof: From theorem 2.3

$$\mu_j = 1, \quad j = k = 1, 2, \dots, r, \quad r \leq n \quad (2.5-39)$$

where  $\mu_j$  is the number of  $J_k$  corresponding to the distinct eigenvalue  $\hat{\lambda}_j$ . The necessary and sufficient conditions from theorem 2.3, give

$$\begin{aligned} \text{rank } B^T \gamma_j = \text{rank } \gamma_j = 1, \quad \hat{\lambda}_j \in \mathcal{C} \\ \text{rank } \gamma_j = \text{rank } \gamma_j = 2, \quad \hat{\lambda}_j \in \mathcal{C}^* \end{aligned} \quad (2.5-40)$$

where

$$\gamma_j = \begin{bmatrix} B^T \gamma_j & k \end{bmatrix}, \quad k = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$\gamma_j = \begin{cases} y & , \hat{\lambda}_j \in \mathcal{C} \\ \begin{bmatrix} y^R & y^I \end{bmatrix} & , \hat{\lambda}_j \in \mathcal{C}^* \end{cases} \quad (2.5-41)$$

in which  $y$  is a single foundation vector corresponding to a Jordan block  $J_j$ ;  $y^R, y^I$  are the real and imaginary parts of the foundation vector  $y$  or  $\bar{y}$  (algebraic sign for the imaginary part ignored) corresponding

to a pair of Jordan blocks  $J_j$ ,  $\bar{J}_j$  associated with a pair of complex conjugate eigenvalues  $\hat{\lambda}_j$ ,  $\bar{\lambda}_j$ . Hence, using (2.5-41) in (2.5-40)

$$\text{rank } B^T y_j = \text{rank } B^T y = 1 \quad \text{for all } \hat{\lambda}_j \in \mathcal{C} \quad (2.5-42)$$

and

$$\text{rank } \gamma_j = \text{rank} \begin{bmatrix} [B^T y^R \ B^T y^I] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ B^T y^R \ B^T y^I \end{bmatrix} = 2$$

$$\text{for all } \hat{\lambda}_j \in \mathcal{C}^*$$

or,

$$\text{rank} \begin{bmatrix} -B^T y^I & B^T y^R \\ B^T y^R & B^T y^I \end{bmatrix} = 2 \quad \text{for all } \hat{\lambda}_j \in \mathcal{C}^* \quad (2.5-43)$$

For real  $\hat{\lambda}_j$ ,  $B^T y$  is an  $m \times 1$  vector, and hence

$$\text{rank } B^T y \leq 1 \\ = 0$$

if and only if the vector  $B^T y = 0$ . But from (2.5-11), this means

$$S_j = e^{-\hat{\lambda}_j t} u^T B^T y_j = 0 \quad \text{for some } \hat{\lambda}_j \in \mathcal{C} \quad (2.5-44)$$

And,  $S_j$  is an element of  $\alpha(t)$ . Thus,

$$\text{rank } B^T y < 1 \quad \text{for some } \hat{\lambda}_j \in G \quad (2.5-45)$$

if and only if a component of  $\alpha(t)$  is zero. For complex  $\hat{\lambda}_j$ , the two vectors,  $2 \times 1$ ,

$$\begin{bmatrix} -B^T y^I \\ B^T y^R \end{bmatrix}, \quad \begin{bmatrix} B^T y^R \\ B^T y^I \end{bmatrix}$$

are L.I. if any one partition  $B^T y^R$  or  $B^T y^I$  is not zero, i.e., these are linearly dependent if and only if  $B^T y^R = B^T y^I = 0$  (this can be proved easily or observed directly). Or,

$$\text{rank} \begin{bmatrix} -B^T y^I & B^T y^R \\ B^T y^R & B^T y^I \end{bmatrix} < 2 \quad \text{for some } \hat{\lambda}_j \in G^*$$

if and only if  $B^T y^R = B^T y^I = 0$ , i.e.,  $\gamma_j = 0$  for some  $\hat{\lambda}_j \in G^*$ .

But from (2.5-14) this implies

$$S_j = 2 e^{\hat{\lambda}_j t} [u_a^T \ u_b^T] \gamma_j = 0 \quad \text{for some } \hat{\lambda}_j \in G^* \quad (2.5-46)$$

And  $S_j$  is a pair of elements of  $\alpha(t)$ . Thus,

$$\text{rank } \gamma_j < 2 \quad \text{for some } \hat{\lambda}_j \in G^* \quad (2.5-47)$$

if and only if a pair of components of  $\alpha(t)$  is zero.

This shows that for all  $\hat{\lambda}_j$ , real or complex, the ranks of  $B^T \gamma_j$  or  $\gamma_j$  are full if there are no zero elements in  $\alpha(t)$ , and full rank for these matrices is a sufficient condition (theorem 2.3) for complete controllability of the system. This proves the theorem.

In this section we have discussed the spatial form from controllability point of view, and proved a few useful results. Now, we are ready to proceed with the construction of an approximation for the recovery region  $\mathcal{R}$ .

## 2.6 Approximation for Recovery Region

In section 2.3 we discussed the recovery region  $\mathcal{R}$  and some approximations for 2 dimensional systems (Fig. 1,2,3). The approximation was in the form of a parallelogram enclosing  $\mathcal{R}$ , and thus forming an upper bound for  $\mathcal{R}$ . Here, we will extend the parallelogram concept to the  $n$  dimensional space and obtain an  $n$  dimensional parallelopiped. The  $n$  directions needed to span the space come from the columns  $c_j$  of the  $C$  matrix in the spatial form for  $\delta$  (see(2.4-25)). Thus, a set of  $n$  vectors in these directions will form the semiaxes of the  $n$ -D ( $D$  for dimensional) parallelopiped. The magnitudes of these vectors are provided by the magnitudes of the  $c_j$  vectors as well as by the components of  $\alpha(t)$ . Since we seek an upper bound of the region for a finite time  $T$  with bounded controls, we will maximize each time integral component of  $\alpha(t)$  for the period  $T$  using the upper bound for all the controls  $u_i$ ,  $i=1,2,\dots,m$ , i.e.,  $|u_i| = 1$ . Thus, each semiaxis of the  $n$ -D parallelopiped represents the maximum component a state space displacement can have in that direction in order that it can be brought to zero in time  $T$ . We mentioned in section 2.3, for 2 dimensional case that at least one point in the recovery region  $\mathcal{R}$  lies on every side of the parallelogram. Similarly, for the  $n$  dimensional case at least one point in  $\mathcal{R}$  will lie on every surface of the  $n$ -D parallelopiped each of whose surfaces consists of  $n-1$  independent vectors as edges which are parallel to  $n-1$  of the  $n$  semiaxes. The semiaxis that does not have a parallel vector in the surface is the one on whose tip this surface rests (semiaxis is from the

origin to the center of a surface). Of course, an  $n$ -D parallelopiped has  $n$  pairs of parallel  $n-1$  dimensional surfaces.

Let  $\mathcal{Q}^*$  represent the approximation, the  $n$ -D parallelopiped, to the recovery region  $\mathcal{Q}$ , which is to be constructed from the equation (spatial form (2.4-26))

$$\begin{aligned} \delta &= C \int_0^T \alpha(t) dt \\ &= [c_1 \ c_2 \ \dots \ c_n] \begin{bmatrix} \int_0^T \alpha_1(t) dt \\ \int_0^T \alpha_2(t) dt \\ \vdots \\ \int_0^T \alpha_n(t) dt \end{bmatrix} \end{aligned} \quad (2.6-1)$$

Define

$$\begin{aligned} \gamma^T &= [\gamma_1 \ \gamma_2 \ \dots \ \gamma_n] \\ D_\gamma &= \text{diag} [\gamma_1, \gamma_2, \dots, \gamma_n] \\ \gamma_j &= \max \int_0^T \alpha_j(t) dt, \quad t \in [0, T], \quad |\alpha_i| = 1, \\ &\quad i=1, 2, \dots, m; \quad j=1, 2, \dots, n \end{aligned} \quad (2.6-2)$$

In terms of these notations  $\mathcal{Q}^*$  can be denoted by a set of  $n$  vectors  $\gamma_j c_j$ ,  $j=1, 2, \dots, n$  in matrix form, these vectors forming the semiaxes of the  $n$ -D parallelopiped. Or,

$$\mathcal{Q}^* = [\gamma_1 c_1, \gamma_2 c_2, \dots, \gamma_n c_n]$$

$$= [c_1 \ c_2 \ \dots \ c_n] \text{diag} [1, 1, \dots, 1] \quad (2.6-3)$$

$$= C D_Y$$

To obtain an approximation to the degree of controllability  $\rho$  of the recovery region  $R$  we have to get the minimum of the distances from the origin to all the surfaces of the  $n$ -D parallelopiped. The distance from the origin to any surface is measured along the normal to that surface from the origin. Then the normal distance from the origin to a surface of the  $n$ -D parallelopiped is the component of its tip vector<sup>†</sup> along the normal to this surface. In the following theorem it is shown how to obtain the normal distances to the surfaces of an  $n$ -D parallelopiped.

Theorem 2.5: Let  $F = [a_1 \ a_2 \ \dots \ a_n]$  prescribe an  $n$ -D parallelopiped in real space whose semiaxes are given by the linearly independent  $n \times 1$  columns  $a_j$ ,  $j=1,2,\dots,n$ . Then the normal distances  $d_j$  to the surfaces of this parallelopiped are given by the reciprocals of the magnitudes of the column vectors of  $(F^T)^{-1}$ .

Proof: Let  $(F^T)^{-1} = [\bar{a}_1 \ \bar{a}_2 \ \dots \ \bar{a}_n]$  in which  $\bar{a}_j$ ,  $j=1,2,\dots,n$  are  $n \times 1$  columns. Since  $F^{-1}F = E$ , the unit matrix, we have

$$F^{-1}F = \begin{bmatrix} \bar{a}_1^T \\ \bar{a}_2^T \\ \vdots \\ \bar{a}_n^T \end{bmatrix} [a_1 \ a_2 \ \dots \ a_n] = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

(2.6-4)

<sup>†</sup>A tip vector of a surface is the semiaxis on whose end the  $n-1$  dimensional surface rests. This surface does not have edges parallel to its tip vector.

$$O_T, \quad \begin{bmatrix} \tilde{a}_1^T a_1 & \tilde{a}_1^T a_2 & \dots & \tilde{a}_1^T a_n \\ \tilde{a}_2^T a_1 & \tilde{a}_2^T a_2 & \dots & \tilde{a}_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_n^T a_1 & \dots & \dots & \tilde{a}_n^T a_n \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

from which

$$\tilde{a}_j^T a_k = \delta_{jk}, \quad j = 1, 2, \dots, n, \quad k = 1, 2, \dots, n \quad (2.5-5)$$

where  $\delta_{jk}$  is the Kronecker delta. Equation (2.5-5) is the restatement of the fact that  $\tilde{a}_j$  is orthogonal to all  $a_k$ ,  $k \neq j$ . In an  $n$  dimensional parallelepiped a normal to a surface is orthogonal to  $n-1$  vectors of which the surface is composed of. Or, a normal is orthogonal to  $n-1$  of the semiaxes. Since  $\tilde{a}_j$  is orthogonal to  $n-1$  of the  $a_k$ 's, it represents the normal direction to the surface whose edges are parallel to  $a_k$  for all  $k$  except  $k=j$ . The tip vector for this surface is  $a_j$ , and the normal distance  $d_j$  from the origin to this surface is the component of  $a_j$  along the normal direction given by  $\tilde{a}_j$ . Or,

$$d_j = \frac{(a_j, \tilde{a}_j)}{\sqrt{(\tilde{a}_j, \tilde{a}_j)}} = \frac{\tilde{a}_j^T a_j}{\|\tilde{a}_j\|}, \quad j = 1, 2, \dots, n \quad (2.5-6)$$



and using (2.6-5)

$$d_j = \frac{1}{\|\tilde{a}_j\|}, \quad j=1, 2, \dots, n \quad (2.6-7)$$

where  $\|\tilde{a}_j\|$  is the magnitude (Euclidean norm) of the column vector  $\tilde{a}_j$  of  $(F^T)^{-1}$ . This proves the theorem.

Now, from (2.6-3),

$$[(R^*)^T]^{-1} = [(C D_Y)^T]^{-1} = (C^T)^{-1} D_Y^{-1} \quad (2.6-8)$$

Let

$$(C^T)^{-1} = [g_1 \ g_2 \ \dots \ g_n]$$

$$g_j = \|g_j\| \hat{g}_j, \quad \|g_j\| = \sqrt{g_j^T g_j}, \quad (2.6-9)$$

$$j=1, 2, \dots, n$$

where  $\hat{g}_j$  denote unit vectors. Then

$$[(R^*)^T]^{-1} = [\hat{g}_1 \ \hat{g}_2 \ \dots \ \hat{g}_n] \text{diag}[\|g_1\|, \|g_2\|, \dots, \|g_n\|] \text{diag}\left[\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \dots, \frac{1}{\gamma_n}\right]$$

$$= [\hat{g}_1 \ \hat{g}_2 \ \dots \ \hat{g}_n] \text{diag}\left[\frac{\|g_1\|}{\gamma_1}, \frac{\|g_2\|}{\gamma_2}, \dots, \frac{\|g_n\|}{\gamma_n}\right] \quad (2.6-10)$$

Using theorem 2.5, the normal distances from the origin to the surfaces of the region  $R^*$  are given by

$$d_j = \frac{1}{\|g_j\|}, \quad j=1, 2, \dots, n \quad (2.6-11)$$

The approximation  $\rho^*$  of the degree of controllability  $\rho$  from

the parallelepiped bound  $\rho^*$  on the recovery region  $\mathcal{R}$  is

$$\rho^* = \min_j d_j, \quad j = 1, 2, \dots, n. \quad (2.6-12)$$

This approximate degree of controllability can be made arbitrarily tight by including additional directions in the  $n$  dimensional space. Suppose  $e$  is any desired unit vector in one such direction, then maximize

$$\int_0^T e^T C x(t) dt \quad (2.6-13)$$

over the period  $T$  with  $|u_i| = 1$ ,  $i = 1, 2, \dots, m$ . This quantity represents the maximum component in the direction  $e$  of a state in the recovery region  $\mathcal{R}$ , and hence no points in  $\mathcal{R}$  lie beyond the surface orthogonal to  $e$  and a distance given by (2.6-13). This is similar to what we discussed in section 2.3 for a two dimensional state space. By considering a set of such  $e$  directions a set of distances (2.6-13) is obtained. Let  $\hat{\rho}$  be the minimum of these distances. Then an improved estimate of the degree of controllability is  $\rho^{**} = \min(\rho^*, \hat{\rho})$ , and  $\rho^{**} > \rho$  can be made arbitrarily close to the true degree of controllability  $\rho$  by picking a sufficient number of directions  $e$ .

The approximate degree of controllability  $\rho^*$  in (2.6-12) will go to zero when the system becomes uncontrollable as indicated in the following theorem.

**Theorem 2.6:** Suppose  $J$  is the Jordan canonical form of  $A$  in the system (2.5-19) and  $J_k$  the Jordan blocks associated with the eigenvalues  $\lambda_k$ ,  $k = 1, 2, \dots, r$ ,  $r \leq n$ . If the eigenvalues  $\lambda_k$  are

distinct then the approximate degree of controllability  $\rho^*$  is zero if and only if the degree of controllability  $\rho$  is zero, i.e.,  $\rho^*$  based on the minimum normal distance to the surface of the parallelopiped  $\mathcal{Q}^*$  will be zero if and only if the system is uncontrollable.

Proof: The normal distances to the surfaces of the parallelopiped  $\mathcal{Q}^*$ , from (2.6-11), are

$$d_j = \frac{\gamma_j}{\|g_j\|}, \quad j = 1, 2, \dots, n$$

where  $\gamma_j$  are defined in (2.6-2). Since  $\rho^* = \min_j d_j$ , it will be zero if and only if some  $d_j = 0$ , but  $d_j = 0$  if and only if  $\gamma_j = 0$ . From (2.6-2)

$$\gamma_j = \max_0^T \int_0^T \alpha_j(t) dt, \quad t \in [0, T], \quad |u_i| = 1, \quad i = 1, 2, \dots, m$$

and  $\gamma_j = 0$  if and only if  $\alpha_j = 0$ . Thus,  $\rho^* = 0$  if and only if some  $\alpha_j = 0$ . From theorem 2.4 the system is uncontrollable if and only if there exist one or more zero components in  $\alpha(t)$ , i.e., some  $\alpha_j(t) = 0$ . Hence,  $\rho^* = 0$  if and only if the system is uncontrollable. This proves the theorem.

When two or more Jordan blocks are associated with the same eigenvalue some modification is necessary so that  $\rho^*$  will be zero if and only if the system is uncontrollable. We adopt what is known as singular value decomposition to modify the foundation elements corresponding to the set of Jordan blocks associated with one eigenvalue. We denote such a set as  $S_j$  (see (2.5-10)) and will use

the same terminology for all quantities unless otherwise mentioned. Similar to the approach adopted in transforming complex vectors in the spatial form to real vectors, we will group the vectors of  $\delta$  that contain the elements of  $S_j$  and transform this set into a similar set so that the same process of replacement in the spatial form can be adopted. The new elements of  $\alpha(t)$  will give the property desired. We will illustrate the process for real eigenvalues, and for complex eigenvalues it is exactly similar.

Consider a group of vectors (a summation) of  $\delta$  associated with  $S_j$  corresponding to the eigenvalue  $\hat{\lambda}_j$ , i.e.,

$$\sum_{i=1}^{M_j} c_i \int_0^T \alpha_i(t) dt \quad (2.5-14)$$

where the  $\alpha_i(t)$  belong to  $S_j$  (in column form it is denoted by  $S_j^T$ ). Denoting the set of  $c_i$  by  $C_{M_j}$  ( $M_j \times 1$  matrix) and substituting for  $S_j$  from (2.5-11) this equation becomes

$$\begin{aligned} \sum_{i=1}^{M_j} c_i \int_0^T \alpha_i(t) dt &= \sum_{i=1}^{M_j} c_i \int_0^T S_j^T dt = [c_1 \ c_2 \ \dots \ c_{M_j}] \int_0^T e^{-\hat{\lambda}_j t} Y_j^T B u dt \\ &= C_{M_j} Y_j^T B \int_0^T e^{-\hat{\lambda}_j t} u dt \end{aligned} \quad (2.6-15)$$

Since the necessary and sufficient condition is related to the rank of the matrix  $B^T Y_j$  (see theorem 2.3) this is to be modified by singular value decomposition technique. Details of this technique can be found

in many texts on Linear Algebra, e.g., reference [4], and here we will just use the results for our purpose. By this technique the matrix  $Y_j^T B$  can be written as

$$Y_j^T B = L_j W_j Z_j \quad (2.6-15)$$

in which  $L_j$ ,  $Z_j$  are orthogonal matrices of order  $\mu_j$ ,  $m$ , respectively, and  $W_j$  is a  $\mu_j \times m$  matrix of the form

$$W_j = \begin{bmatrix} \Lambda_j & 0 \\ 0 & 0 \end{bmatrix} \quad (2.6-17)$$

The  $\Lambda_j$  is a diagonal matrix consisting of as many nonzero values called singular values as the rank of  $W_j$ . Substituting (2.6-15) in (2.6-15) we have

$$\begin{aligned} C_{\mu_j} Y_j^T B \int_0^T e^{-\hat{\lambda}_j t} u dt &= C_{\mu_j} L_j \int_0^T e^{-\hat{\lambda}_j t} W_j Z_j u dt \\ &= \hat{C}_{\mu_j} \int_0^T e^{-\hat{\lambda}_j t} W_j Z_j u dt \end{aligned} \quad (2.6-18)$$

where  $\hat{C}_{\mu_j} = C_{\mu_j} L_j$  which is a set of  $\mu_j$  L.I. vectors. Now, let

$$\hat{S}_j^T = e^{-\hat{\lambda}_j t} W_j Z_j u \quad (2.6-19)$$

denote the  $\mu_j$  elements  $\hat{\alpha}_i(t)$ ,  $i=1,2,\dots,\mu_j$ . With this (2.6-18) can be written as

$$\hat{C}_{\mu_j} \int_0^T \hat{S}_j^T dt \quad (2.6-20)$$

This form is ready for replacement in the  $\delta$  equation, i.e., replace  $C_{\mu_j}$  by  $\hat{C}_{\mu_j}$  and  $\alpha_i(t)$  by  $\hat{\alpha}_i(t)$ . If  $\mu_j$  is of order  $l_j$  (which indicates the rank of  $W_j$  is  $l_j$ ) then expanding  $W_j Z_j$  we will get

$$\hat{S}_j = \left[ \underbrace{\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_{l_j}}_{l_j}, \underbrace{0, 0, \dots, 0}_{\mu_j - l_j} \right] \quad (2.6-21)$$

This has  $\mu_j - l_j$  zeroes for the  $\hat{\alpha}_i(t)$  coefficients. But  $l_j$  being the rank of  $Y_j^T$  if it is less than  $\mu_j$  the system is uncontrollable. In other words, if there is a zero element present in  $\hat{S}_j$  then the system is uncontrollable. On the other hand, if  $l_j$  is equal to  $\mu_j$ ,  $Y_j^T$  is of full rank which satisfies the sufficient condition for complete controllability as far as this eigenvalue  $\hat{\lambda}_j$  is concerned. Hence, the coefficients  $\hat{\alpha}_i(t)$  are independent. Thus, only a zero indicates dependency of  $\hat{\alpha}_i(t)$ . This process is repeated for all distinct real eigenvalues  $\hat{\lambda}_j$ .

For complex eigenvalues the same procedure is adopted. The equivalent of (2.6-15) is

$$C_{2\mu_j} \gamma_j^T \int_0^T e^{2\hat{\lambda}_j t} \begin{bmatrix} u_a \\ u_b \end{bmatrix} dt \quad (2.6-22)$$

where  $C_{2\mu_j} = [c_1, c_2, \dots, c_{2\mu_j}]$ , an  $n \times 2\mu_j$  matrix. Again  $\gamma_j^T$  is decomposed similar to (2.6-16) (note  $Y_j$  is a  $n \times 2\mu_j$  matrix), i.e.,

$$\gamma_j^T = \begin{bmatrix} K^T \gamma_j^T B & \gamma_j^T B \end{bmatrix} = L_j W_j Z_j \quad (2.6-23)$$

Hence, (2.6-22) becomes

$$C_{2\mu_j} L_j \int_0^T e^{z_j^* t} W_j z_j \begin{bmatrix} u_a \\ u_b \end{bmatrix} dt = \hat{C}_{2\mu_j} \int_0^T \hat{S}_j^T dt \quad (2.6-24)$$

where  $\hat{C}_{2\mu_j} = C_{2\mu_j} L_j$ , a set of  $2\mu_j$  L.I. vectors, and

$$\hat{S}_j^T = e^{z_j^* t} W_j z_j \begin{bmatrix} u_a \\ u_b \end{bmatrix} \quad (2.6-25)$$

denotes the  $2\mu_j$  elements  $\hat{\alpha}_i(t)$ ,  $i=1,2,\dots,2\mu_j$ . The form in (2.6-24) is ready for replacement in the  $\delta$  equation as discussed earlier. Now, suppose  $W_j$  is of rank  $l_j$  then  $\text{rank } \gamma_j = l_j$ . And if  $l_j < 2\mu_j$  then there will be  $2\mu_j - l_j$  zeroes of  $\hat{\alpha}_i(t)$  in  $\hat{S}_j^T$ . The system, of course, is uncontrollable ( $\text{rank } \gamma_j < 2\mu_j$ ). On the other hand, if  $l_j = 2\mu_j$  the system is completely controllable ( $\text{rank } \gamma_j = 2\mu_j$ ) as far as this eigenvalue  $\hat{\lambda}_j$  is concerned. Hence, all  $2\mu_j$  coefficients  $\hat{\alpha}_i(t)$  are independent. Thus, only the presence of zeroes indicates dependency of  $\hat{\alpha}_i(t)$ . This process is repeated for all distinct complex eigenvalues  $\hat{\lambda}_j$  or  $\bar{\hat{\lambda}}_j$ .

In the above a modification is described to be adopted in the spatial form of  $\delta$  for the columns in  $C$  and the elements in  $\alpha(t)$  in the case of more than one Jordan blocks associated with one eigenvalue. This involves replacing a set of  $c_j$  in  $C$  by another L.I. set  $\hat{c}_j$  and a corresponding replacement of  $\alpha_j(t)$  in  $\alpha(t)$  by  $\hat{\alpha}_j(t)$  which correspond to the foundation elements associated with an eigenvalue. We showed that the system is uncontrollable if and only if an element  $\hat{\alpha}_j(t)$

goes to zero. Combining this with theorem 2.6 and following the same line of proof we conclude that the approximate degree of controllability  $\rho^*$  goes to zero if and only if the system is uncontrollable. This can be summarized in the following theorem.

Theorem 2.7: For a linear time invariant system (2.5-19) the approximate degree of controllability  $\rho^*$  based on the parallelepiped region  $R^*$  goes to zero if and only if the system is uncontrollable.

If there are more than one Jordan blocks associated with an eigenvalue apply modification based on singular value decomposition as described above to the sets of columns of  $C$  and foundation elements of  $q(t)$  associated with sets of Jordan blocks corresponding to any distinct eigenvalue. The rest of the proof is as given in theorem 2.6.



## 2.7 Summary

In this chapter attention was directed to general linear time invariant systems. We started with the question of how effective is a distribution of actuators on a very large flexible spacecraft in controlling its attitude and shape? To comprehend the meaning of "effective" we sought to develop a concept of the degree of controllability. This development followed a rational approach first by showing the unsuitability of certain candidates to serve as a definition for the degree of controllability, and then resulting in a meaningful definition which would account for all the pertinent factors such as controllability, total time, control effort, stability, and control objective that will have a bearing on the degree of controllability. Once a definition was formulated suitable approximations had to be developed for the recovery region and the degree of controllability so that this definition is easily applicable to real problems and numerically manageable, the approximations approaching the true values as the computational effort is increased. The mathematical approach adopted, besides leading us to our desired goal, showed the system equations in special form thereby enabling us to derive some relatively simple tests for complete controllability of the system.

Thus, in short, the emphasis in this chapter has been development of a rational concept to obtain a meaningful definition for the degree of controllability and taking it to the stage of usefulness from an applications point of view.

### 3. ANALYSIS OF LARGE FLEXIBLE SPACECRAFT

#### 3.1 Introduction

Having developed the necessary concepts for the degree of controllability of a system we are in a position to apply these to our fundamental problem, the study of the effect of location of actuators on very large flexible spacecraft. To do this first we have to obtain the state space form (2.2-2) for the system dynamics equations of motion for the spacecraft. The following are the essential steps involved in obtaining the state space form.

1. Choosing a general model for a typical very large flexible spacecraft.
2. Derivation of equations of motion and their linearization about the equilibrium state.
3. Translating the generalized forces into the normalized effort  $u$ , and introducing weighting factors for the coordinates to obtain normalized state variables.

In this chapter we will be concerned with the analysis of a general model for a large flexible spacecraft and obtaining the state space form for the dynamic equations describing the system.

### 3.2 Typical Planar Motion Model of a Very Large Flexible Spacecraft

The typical planar model consists of essentially a rigid body  $B_R$ , and a flexible body  $B_F$ , with the rigid body as a central core separating the flexible body. Thus, the flexible body consists of two very large appendages attached to the rigid body one on either side as shown in Fig. 4. The system is considered inertially at rest, i.e., any disturbance to the system results in some deformation of  $B_F$  and some motion of  $B_R$  such that the center of mass of the system remains stationary in inertial space. The system will be treated as a continuum and the hybrid coordinate approach [5] will be used. The flexible body  $B_F$  will be treated as elastic beams and the deformation in the plane of rotation will be assumed normal to their undeformed axis.

THIS PAGE IS  
OF POOR QUALITY

### 3.3 Dynamics

The following notations refer to the model shown in Fig. 4 .

- O: Center of mass of the system stationary in inertial space.
- C: Point fixed in  $B_R$  with which O coincides when there is no deformation of the flexible body  $B_F$  .
- B: Frame of cartesian body axes  $\hat{b}_1, \hat{b}_2, \hat{b}_3$  whose origin is at C. The plane of motion is  $\hat{b}_1 - \hat{b}_2$  , the axis of the undeformed flexible appendages is parallel to  $\hat{b}_1$  , and their deformation is in  $\hat{b}_2$  direction.
- $m_R, m_F, m_S$ : Masses of the rigid body ( $B_R$ ), the flexible body ( $B_F$ ), and the system ( $B_R + B_F$ ), respectively.
- dm: Differential mass element in the system.
- R: Position vector of a mass element in the system in the inertial frame (i.e., from O)
- $\underline{R}_c$ : Vector drawn from the center of mass O to the point C.
- $\underline{r}'$ : Position vector of any mass element in the system with respect to C.
- $\underline{r}_n$ : Vector from C normal to the undeformed axis of  $B_F$  .
- $\underline{r}_a$ : Vector in  $\hat{b}_1$  direction denoting the offset of the flexible component from the base of  $\underline{r}_n$  .
- $\underline{r}$ :  $\hat{b}_1$  component of  $\underline{r}'$  for mass elements of  $B_F$  .
- $\underline{y}(\underline{r}, t)$ : Displacement of mass elements of  $B_F$  relative to its undeformed axis.
- $\Theta$ : Inertial attitude of  $B_R$  (and rotation of body frame B).

$$\left. \begin{array}{l} \dot{I}_n(*): \\ \dot{I}_{nd}(*): \\ \frac{d}{dt} \dot{s}(*): \end{array} \right\} \text{Time derivative of } (*) \text{ with respect to the inertial frame.}$$

$$\left. \begin{array}{l} \dot{s}(*): \\ \frac{d}{dt} \dot{s}(*): \end{array} \right\} \text{Time derivative of } (*) \text{ with respect to the body frame B.}$$

From these we have the following relations:

$$\underline{R} = \underline{R}_c + \underline{r}'; \quad m_s = m_R + m_F; \quad \underline{r}' = \underline{r}_n + \underline{r} + \underline{y}(\underline{r}, t) \quad \text{for } B_F \quad (3.3-1)$$

Now, let

$$\underline{u} = \underline{u} \hat{b}_3; \quad \underline{y}(\underline{r}, t) = y(\underline{r}, t) \hat{b}_2; \quad \underline{r}' = r_1' \hat{b}_1 + r_2' \hat{b}_2$$

$$\underline{r}_n = r_n \hat{b}_2; \quad \underline{r} = r \hat{b}_1 \quad (3.3-2)$$

so that

$$\underline{r}' = r_n \hat{b}_2 + r \hat{b}_1 + y \hat{b}_2 = r \hat{b}_1 + (r_n + y) \hat{b}_2 \quad \text{for } B_F \quad (3.3-3)$$

From these three equations, we can write

$$r_1' = \begin{cases} r_1' & \text{for } B_R \\ r & \text{for } B_F \end{cases}; \quad r_2' = \begin{cases} r_2' & \text{for } B_R \\ r_n + y & \text{for } B_F \end{cases}$$

$$\theta \dot{r}_1' = 0 \quad \text{for } B_R \text{ and } B_F; \quad \theta \dot{r}_2' = \begin{cases} 0 & \text{for } B_R \\ \dot{y} & \text{for } B_F \end{cases} \quad (3.3-4)$$

The expression for  $\underline{R}_c$  can be obtained by considering the center of mass of the system with respect to the body frame B as follows:

$$\begin{aligned} -m_s \underline{R}_c &= \int_{\text{system}} \underline{r}' dm = \int_{B_R} \underline{r}' dm + \int_{B_F} \underline{r}' dm \\ &= \int_{B_R} \underline{r}' dm + \int_{B_F} (r_n + r) dm + \int_{B_F} y(\underline{r}, t) dm \end{aligned} \quad (3.3-5)$$

where (3.3-1) has been used for  $\underline{r}'$  for  $B_F$ . But when there is no deformation of  $B_F$ ,  $\underline{R}_c = 0$  (see definition of C), i.e.,

$$\int_{B_F} \underline{r}' dm + \int_{B_F} (\underline{r}_n + \underline{r}) dm = 0 \quad (3.3-6)$$

Using this in (3.3-5) we obtain

$$\underline{R}_c = -\frac{1}{m_S} \int_{B_F} \underline{y}(\underline{r}, t) dm = \left[ -\frac{1}{m_S} \int_{B_F} \underline{y}(\underline{r}, t) dm \right] \hat{b}_2 \quad (3.3-7)$$

where  $\underline{y}$  has been substituted from (3.3-2). Let

$$\underline{R}_c = -\frac{1}{m_S} \int_{B_F} \underline{y}(\underline{r}, t) dm, \text{ so that } \underline{R}_c = R_c \hat{b}_2 \quad (3.3-8)$$

Hence, from (3.3-1), (3.3-2)

$$\underline{R} = \underline{R}_c + \underline{r}' = R_c \hat{b}_2 + \gamma_1' \hat{b}_1 + \gamma_2' \hat{b}_2 = \gamma_1' \hat{b}_1 + (R_c + \gamma_2') \hat{b}_2 \quad (3.3-9)$$

Now,

$$\begin{aligned} {}^I \dot{\underline{R}} &= {}^0 \dot{\underline{R}} + \dot{\underline{H}} \times \underline{R} = {}^B \dot{\gamma}_1' \hat{b}_1 + ({}^0 \dot{R}_c + {}^B \dot{\gamma}_2') \hat{b}_2 \\ &\quad + \dot{\underline{H}} \hat{b}_2 \times [\gamma_1' \hat{b}_1 + (R_c + \gamma_2') \hat{b}_2] \quad (3.3-10) \\ &= -\dot{\underline{H}} (R_c + \gamma_2') \hat{b}_1 + ({}^0 \dot{R}_c + {}^B \dot{\gamma}_2' + \dot{\underline{H}} \gamma_1') \hat{b}_2 \end{aligned}$$

where  ${}^B \dot{\gamma}_1' = 0$  from (3.3-4). Hence,

$$\begin{aligned} {}^I \dot{\underline{R}} \cdot {}^I \dot{\underline{R}} &= \dot{\underline{H}}^2 [R_c^2 + (\gamma_1')^2 + (\gamma_2')^2 + 2 R_c \gamma_2'] + {}^0 \dot{R}_c^2 + ({}^B \dot{\gamma}_2')^2 \\ &\quad + 2 ({}^B \dot{R}_c) ({}^B \dot{\gamma}_2') + 2 \dot{\underline{H}} \gamma_1' ({}^B \dot{\gamma}_2') + 2 \dot{\underline{H}} \gamma_1' ({}^0 \dot{R}_c) \quad (3.3-11) \end{aligned}$$



Define the following system constants:

$$\begin{aligned}
 I_S &= \int_{B_R} (\dot{r}')^2 dm + \int_{B_F} r^2 dm + m_F r_n^2 \\
 C_{S1} &= \int_{B_R} r'_1 dm + m_F r_n \\
 C_{S2} &= \int_{B_R} r'_1 dm + \int_{B_F} r dm = \int_{\text{system}} r'_1 dm \quad (3.3-15)
 \end{aligned}$$

in which  $I_S$  represents the system moment of inertia about point C when there is no deformation of  $B_F$  ( $y = 0$ ). Substituting (3.3-14) in (3.3-13) and using (3.3-15) we obtain

$$\begin{aligned}
 T_S &= \dot{\Theta}^2 \left[ \frac{1}{2} I_S + \frac{1}{2} m_S R_c^2 + r_n \int_{B_F} y dm + C_{S1} R_c + \frac{1}{2} \int_{B_F} y^2 dm + R_c \int_{B_F} y dm \right] \\
 &+ \dot{\Theta} \left[ C_{S2} (\dot{\Theta} R_c) + \int_{B_F} r \dot{y} dm \right] + \frac{1}{2} m_S (\dot{\Theta} R_c)^2 + \dot{\Theta} R_c \int_{B_F} \dot{y} dm \\
 &+ \frac{1}{2} \int_{B_F} \dot{y}^2 dm \quad (3.3-16)
 \end{aligned}$$

where  $m_S = \int_{\text{system}} dm$ .

At this point modal analysis will be adopted for the appendages.

Define

$$\begin{aligned}
 y(r, t) &= \Phi(r) \eta(t) \\
 \Phi(r) &= [\phi_1(r), \phi_2(r), \dots, \phi_N(r)] \\
 \eta^T(t) &= [\eta_1(t), \eta_2(t), \dots, \eta_N(t)] \quad (3.3-17)
 \end{aligned}$$

where  $\phi_j, \eta_j, j=1, 2, \dots, N$ , are scalars, which are functions of space and time, respectively. With this definition we have



$$\begin{aligned}
\int_{B_F} y dm &= \left[ \int_{B_F} \Phi(r) dm \right] \eta(t) ; \int_{B_F} y^2 dm = \eta^T(t) \left[ \int_{B_F} \Phi^T(r) \Phi(r) dm \right] \eta(t) \\
\int_{B_F} \dot{y} dm &= \left[ \int_{B_F} \Phi(r) dm \right] \dot{\eta}(t) ; \int_{B_F} \dot{y}^2 dm = \dot{\eta}^T(t) \left[ \int_{B_F} \Phi^T(r) \Phi(r) dm \right] \dot{\eta}(t) \\
\int_{B_F} r \dot{y} dm &= \left[ \int_{B_F} r \Phi(r) dm \right] \dot{\eta}(t)
\end{aligned} \tag{3.3-18}$$

and using (3.3-3)

$$\begin{aligned}
R_c &= -\frac{1}{m_s} \int_{B_F} y dm = -\frac{1}{m_s} \left[ \int_{B_F} \Phi(r) dm \right] \eta(t) \\
\dot{R}_c &= -\frac{1}{m_s} \left[ \int_{B_F} \Phi(r) dm \right] \dot{\eta}(t)
\end{aligned} \tag{3.3-19}$$

Also, define

$$\begin{aligned}
M_\Phi^T &= [M_{\Phi_1}, M_{\Phi_2}, \dots, M_{\Phi_N}] = \int_{B_F} \Phi(r) dm \\
\Delta &= \int_{B_F} \Phi^T(r) \Phi(r) dm \\
M_{r\Phi}^T &= [M_{r\Phi_1}, M_{r\Phi_2}, \dots, M_{r\Phi_N}] = \int_{B_F} r \Phi(r) dm
\end{aligned} \tag{3.3-20}$$

Substituting (3.3-18) and (3.3-19) in (3.3-15) and using (3.3-20), the expression for the kinetic energy of the system becomes

$$T_s = \dot{\Theta}^2 \left( \frac{1}{2} I_s + \frac{1}{2} \eta^T M \eta + c_{ss} M_\Phi^T \eta \right) + \dot{\Theta} M^T \dot{\eta} + \frac{1}{2} \dot{\eta}^T M \dot{\eta} \tag{3.3-21}$$

where

$$\begin{aligned}
M &= \Delta - \frac{1}{m_s} M_\phi M_\phi^T \\
\mu &= M_{r\phi} - \frac{c_{s2}}{m_s} M_\phi \\
c_{s3} &= \gamma_n - \frac{c_{s1}}{m_s}
\end{aligned} \tag{3.3-22}$$

### 3.3.2 Potential Energy

Since the system is inertially at rest the potential energy in the system is solely due to the elastic deformation of the flexible body  $B_F$ . Let  $V_s$  be the potential energy in the system, and for elastic beams we can write

$$\begin{aligned}
V_s &= \frac{1}{2} \int_{B_F} (EI) \left( \frac{\partial^2 y(r,t)}{\partial r^2} \right)^2 dr & (3.3-23) \\
&= \frac{1}{2} \int_{B_F} (EI) \left\{ \frac{\partial^2 [\Phi(r) \eta(t)]}{\partial r^2} \right\}^2 dr \\
&= \frac{1}{2} \int_{B_F} (EI) \eta^T(t) [\Phi''(r) \Phi''(r)] \eta(t) dr \\
&= \frac{1}{2} \eta^T \left\{ \int_{B_F} (EI) [\Phi''^T \Phi''] dr \right\} \eta \\
&= \frac{1}{2} \eta^T K \eta & (3.3-24)
\end{aligned}$$

where  $y$  is substituted from (3.3-17),  $\Phi''$  denotes the second derivative of  $\Phi$  with respect to  $r$ ,  $(EI)$  is the stiffness of the body  $B_F$ , and

$$K = \int_{B_F} (EI) [\Phi''^T \Phi''] dr \tag{3.3-25}$$

### 3.3.3 Generalized Forces

To derive the generalized forces we will assume that external forces and couples act at discrete locations throughout the system. While forces will be assumed to act at points couples will be assumed to act on infinitesimal elements so that these can have rotation. Let  ${}^{\text{In}}\dot{\underline{R}}_j$  denote the inertial velocity of a point at  $j$ th location in the system where a force  $\underline{F}_j$  is applied, and let  $\underline{\Omega}_j$  denote the inertial angular velocity of an infinitesimal element at  $j$ th location in the system where a pure couple  $\underline{T}_j$  is applied. If  $Q_j$  denotes the generalized force corresponding to the generalized coordinate  $q(t)$ , then

$$Q_j = \sum_{j \in \mathcal{F}} \underline{F}_j \cdot \frac{\partial \dot{\underline{R}}_j}{\partial \dot{q}} + \sum_{j \in \mathcal{C}} \underline{T}_j \cdot \frac{\partial \underline{\Omega}_j}{\partial \dot{q}} \quad (3.3-26)$$

where  $\mathcal{F}$  stands for a class of forces and  $\mathcal{C}$  for a class of pure couples. The coordinates  $q(t)$  are  $\Theta$  and  $\eta$  in our case. From (3.3-10), we can write

$${}^{\text{In}}\dot{\underline{R}}_j = -\dot{\Theta}(\underline{R}_c + \underline{r}_{1j}) \hat{b}_1 + (\dot{\Theta} \underline{R}_c + \dot{\eta} \underline{r}_{1j} + \dot{\Theta} \underline{r}_{1j}) \hat{b}_2 \quad (3.3-27)$$

in which we note that as the location  $j$  varies  $\underline{r}'_1$ ,  $\underline{r}'_2$  and their derivatives are the only quantities that are affected. The inertial angular velocity  $\underline{\Omega}_j$  can be written as

$$\underline{\Omega}_j = \begin{cases} \dot{\Theta} \hat{b}_j & \text{for } B_R \\ (\dot{\Theta} + \frac{\partial}{\partial t} \alpha_j) \hat{b}_j & \text{for } B_F \end{cases} \quad (3.3-28)$$

where  $\alpha_j$  is the slope at  $j$ th location on  $B_F$  relative to the undeformed axis of  $B_F$ . We have

$$\tan \alpha_j = \left. \frac{\partial y(r,t)}{\partial r} \right|_{r=r_j} = \Phi'(r_j) \eta(t) \quad (3.3-29)$$

where  $\Phi'(r_j)$  denotes the first derivative of  $\Phi$  with respect to  $r$  at  $r = r_j$ . For small deformations  $\tan \alpha_j = \alpha_j$ , and hence

$$\frac{\partial}{\partial t} \alpha_j = \frac{\partial}{\partial t} (\tan \alpha_j) = \Phi'(r_j) \dot{\eta}(t) \quad (3.3-30)$$

which gives

$$\underline{\dot{R}}_j = \begin{cases} \dot{\Theta} \hat{b}_3 & \text{for } \Theta_R \\ [\dot{\Theta} + \Phi'(r_j) \dot{\eta}(t)] \hat{b}_3 & \text{for } \Theta_F \end{cases} \quad (3.3-31)$$

Now, from (3.3-19) and (3.3-20)

$$R_c = -\frac{1}{m_s} \mathcal{M}_\Phi^T \eta \quad ; \quad {}^B \dot{R}_c = -\frac{1}{m_s} \mathcal{M}_\Phi^T \dot{\eta} \quad (3.3-32)$$

and from (3.3-4) and (3.3-17)

$$\begin{aligned} \tau'_{1j} &= \begin{cases} \tau'_{1j} & \text{for } \Theta_R \\ \tau_j & \text{for } \Theta_F \end{cases} ; \quad \tau'_{2j} = \begin{cases} \tau'_{2j} & \text{for } \Theta_R \\ \tau_n + \Phi(r_j) \eta & \text{for } \Theta_F \end{cases} \\ {}^B \dot{\tau}'_{2j} &= \begin{cases} 0 & \text{for } \Theta_R \\ \Phi(r_j) \dot{\eta} & \text{for } \Theta_F \end{cases} \end{aligned} \quad (3.3-33)$$

With these equations, from (3.3-27) and (3.3-31), for  $q = \Theta$  and  $q = \eta$ , we can write

$$\begin{aligned} \frac{\partial \mathcal{J}_j}{\partial \Theta} &= -(\dot{R}_c + \tau'_{2j}) \hat{b}_1 + \tau'_{1j} \hat{b}_2 \\ &= \begin{cases} \left( \frac{1}{m_s} \mathcal{M}_\Phi^T \eta - \tau'_{2j} \right) \hat{b}_1 + \tau'_{1j} \hat{b}_2 & \text{for } \Theta_R \\ \left\{ \left[ \frac{1}{m_s} \mathcal{M}_\Phi^T - \Phi(r_j) \right] \eta - \tau_n \right\} \hat{b}_1 + \tau_j \hat{b}_2 & \text{for } \Theta_F \end{cases} \end{aligned}$$

$$\begin{aligned}
\frac{\partial \hat{R}_j}{\partial \dot{\eta}} &= \frac{\partial}{\partial \dot{\eta}} \left( \hat{R}_c + \hat{\tau}_{2j} + \hat{\tau}_{1j} \right) \hat{b}_2 \\
&= \begin{cases} -\frac{1}{m_s} M_\phi \hat{b}_2 & \text{for } \theta_R \\ \left[ -\frac{1}{m_s} M_\phi + \Phi^T(r_j) \right] \hat{b}_2 & \text{for } \theta_F \end{cases} \\
\frac{\partial \hat{R}_j}{\partial \dot{\theta}} &= \hat{b}_3 \quad ; \quad \frac{\partial \hat{R}_j}{\partial \dot{\eta}} = \begin{cases} 0 & \text{for } \theta_R \\ \Phi'^T(r_j) \hat{b}_3 & \text{for } \theta_F \end{cases} \quad (3.3-34)
\end{aligned}$$

Let

$$\begin{aligned}
F_j &= L_j \hat{b}_1 + \gamma_j \hat{b}_2 \\
\tau_j &= \tau_j \hat{b}_3 \quad (3.3-35)
\end{aligned}$$

The forces and the couples are applied in the plane of motion. The equation for the generalized forces in (3.3-25) then becomes

$$Q_\theta = \begin{cases} \sum_{j \in \mathcal{F}} \left\{ L_j \left( \frac{1}{m_s} M_\phi^T \eta - \tau_{2j} \right) + \gamma_j \tau_{1j} \right\} + \sum_{j \in \mathcal{C}} \tau_j & \text{for } \theta_R \\ \sum_{j \in \mathcal{F}} \left\{ L_j \left[ \frac{1}{m_s} M_\phi^T - \Phi(r_j) \right] \eta - \tau_{1j} \right\} + \gamma_j \tau_j \right\} + \sum_{j \in \mathcal{C}} \tau_j & \text{for } \theta_F \end{cases} \quad (3.3-36)$$

$$Q_\eta = \begin{cases} \sum_{j \in \mathcal{F}} \gamma_j \left( -\frac{1}{m_s} M_\phi \right) & \text{for } \theta_R \\ \sum_{j \in \mathcal{F}} \left[ -\frac{1}{m_s} M_\phi + \Phi^T(r_j) \right] \gamma_j + \sum_{j \in \mathcal{C}} \tau_j \Phi'^T(r_j) & \text{for } \theta_F \end{cases} \quad (3.3-37)$$

We see that  $Q_\theta$  has terms containing  $\eta$  and we desire the generalized forces to be independent of the coordinates. The nature of the problem is such that the deformation is only in  $\hat{b}_2$  direction, and hence it is reasonable to assume applied forces to be in  $\hat{b}_2$  direction only.

Therefore, we will set  $L_j = 0$  so that

$$F_j = \gamma_j \hat{b}_j \quad (3.3-38)$$

With this the final expression for the generalized forces are:

$$\begin{aligned} Q_\Theta &= \sum_{j \in J} \gamma_j \dot{\tau}_{1j} + \sum_{j \in G} T_j \\ Q_\eta &= \sum_{j \in J} [\Phi^T(\tau_j) \delta_{Fj} - \frac{1}{m_s} M \eta] \gamma_j + \sum_{j \in G} \tau_j \Phi'^T(\tau_j) \delta_{Fj} \end{aligned} \quad (3.3-39)$$

where

$$\dot{\tau}_{1j} = \begin{cases} \dot{\tau}_{1j} & \text{for } B_R \\ \dot{\tau}_j & \text{for } B_F \end{cases} ; \quad \delta_{Fj} = \begin{cases} 0 & \text{for } j \in B_R \\ 1 & \text{for } j \in B_F \end{cases} \quad (3.3-40)$$

#### 3.3.4 Equations of Motion

We have derived the kinetic energy  $T_s$ , the potential energy  $V_s$  and the generalized forces  $Q_q$ . Ignoring the damping (this is reasonable for spacecraft), the Euler-Lagrangian equations can be written as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = Q_q, \quad q = \Theta, \eta \quad (3.3-41)$$

and  $L = T_s - V_s$ . From (3.3-21) and (3.3-24) the Lagrangian

$$\begin{aligned} L = T_s - V_s &= \dot{\Theta}^2 \left( \frac{1}{2} I_s + \frac{1}{2} \eta^T M \eta + c_{ss} M \Phi^T \eta \right) + \dot{\Theta} M^T \dot{\eta} \\ &\quad + \frac{1}{2} \dot{\eta}^T M \dot{\eta} - \frac{1}{2} \eta^T K \eta \end{aligned} \quad (3.3-42)$$

from which the equations of motion for the spacecraft are:

$$\ddot{\Theta}(\mathbf{I}_s + 2c_{s3} \mathbf{M}_\phi^T \dot{\eta} + \dot{\eta}^T \mathbf{M} \dot{\eta}) + 2\dot{\Theta}(c_{s3} \mathbf{M}_\phi^T \dot{\eta} + \dot{\eta}^T \mathbf{M} \dot{\eta}) + \mathbf{M}^T \ddot{\eta} = \mathbf{Q}_\Theta$$

$$\mathbf{M} \ddot{\eta} + \mathbf{K} \eta + \mu \ddot{\Theta} - \dot{\Theta}^2 (\mathbf{M} \eta + c_{s3} \mathbf{M}_\phi) = \mathbf{Q}_\eta \quad (3.3-43)$$

where  $\mathbf{M}$ ,  $\mathbf{K}$  are symmetric and  $\mathbf{Q}_\Theta$ ,  $\mathbf{Q}_\eta$  are as given in (3.3-39).

### 3.3.5 Linearized Equations of motion

In deriving the final form for the generalized forces we already made the assumption that the deformations are very small. We will consider arbitrarily small deformations in the appendages so that nonlinear terms in the coordinates  $\eta_j$  of  $\eta$  can be effectively ignored. Also, if the deformations are small the perturbations in the attitude  $\Theta$  are also small. Hence, nonlinear terms in the attitude perturbations as well as terms involving products of deformation coordinates and attitude perturbations can be neglected. Further, since our concern is mainly in returning the system from a disturbed state to its nominal (equilibrium) state we can, without any loss of generality, assume the nominal state to be at rest, and will consider attitude perturbations about a mean value. Therefore, let

$$\Theta = \bar{\Theta} + \theta ; \quad \dot{\bar{\Theta}} = \ddot{\bar{\Theta}} = 0 \quad (3.3-44)$$

where  $\bar{\Theta}$  is the mean value of the attitude and  $\theta$  its perturbation about this mean. From this

$$\dot{\Theta} = \dot{\theta} ; \quad \ddot{\Theta} = \ddot{\theta} ; \quad \dot{\Theta}^2 = \dot{\theta}^2 \quad (3.3-45)$$

Substituting these in (3.3-43) and neglecting nonlinear terms in the coordinates we get the following linearized  $N+1$  equations of motion for the system:

$$\begin{aligned} I_s \ddot{\theta} + \mu^T \ddot{\eta} &= Q_0 \\ M \ddot{\eta} + k \eta + \mu \ddot{\theta} &= Q_1 \end{aligned} \quad (3.3-46)$$

Or, in matrix form

$$\begin{bmatrix} I_s & \mu^T \\ \mu & M \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\eta} \end{bmatrix} + \begin{bmatrix} 0 & 0^T \\ 0 & k \end{bmatrix} \begin{bmatrix} \theta \\ \eta \end{bmatrix} = \begin{bmatrix} Q_0 \\ Q_1 \end{bmatrix} \quad (3.3-47)$$

where  $\underline{0}$  is an  $N \times 1$  zero vector and  $0^T$  is the transpose of  $\underline{0}$ .

If we define the following matrices

$$\tilde{M} = \begin{bmatrix} I_s & \mu^T \\ \mu & M \end{bmatrix}; \quad \tilde{k} = \begin{bmatrix} 0 & 0^T \\ 0 & k \end{bmatrix}; \quad \tilde{Q} = \begin{bmatrix} Q_0 \\ Q_1 \end{bmatrix}; \quad \tilde{\eta} = \begin{bmatrix} \theta \\ \eta \end{bmatrix} \quad (3.3-48)$$

then the equations of motion in (3.3-47) can be expressed as

$$\tilde{M} \ddot{\tilde{\eta}} + \tilde{k} \tilde{\eta} = \tilde{Q} \quad (3.3-49)$$

or

$$\ddot{\tilde{\eta}} + (\tilde{M}^{-1} \tilde{k}) \tilde{\eta} = \tilde{M}^{-1} \tilde{Q} \quad (3.3-50)$$

which describe the system in the hybrid coordinates  $\theta, \eta$ . We could now proceed to put these equations in state space form. But, since these equations have some special property we will investigate an alternative form for these equations. Note that the matrices  $\tilde{M}$ ,  $\tilde{K}$  are real symmetric and  $\tilde{M}$  is positive definite (associated with kinetic energy). Such a pair of matrices can be diagonalized simultaneously by a principal matrix. Often it is desirable to work in these normal coordinates so we will derive the system equations in orthogonal form.



Let  $\tilde{V} = [\tilde{V}_0 \tilde{V}_1 \tilde{V}_2 \dots \tilde{V}_N]$  be the principal matrix whose columns  $\tilde{V}_i$ ,  $i=0,1,2,\dots,N$ , are of dimension  $N+1$  such that

$$\tilde{V}^T \tilde{M} \tilde{V} = E \quad ; \quad \tilde{V}^T \tilde{K} \tilde{V} = \Lambda \quad (3.3-51)$$

where  $E$  is the unit matrix of order  $N+1$  and  $\Lambda$  is a real diagonal matrix of order  $N+1$ . Details of such transformation can be found in texts on matrix theory, e.g., [6]. Hence, a new set of coordinates  $\xi^T = [\xi_0 \xi_1 \xi_2 \dots \xi_N]$  can be obtained by the transformation

$$\tilde{\eta} = \tilde{V} \xi \quad (3.3-52)$$

Substituting this in (3.3-49) and premultiplying by  $\tilde{V}^T$  and using (3.3-51) we have for the system equations

$$\ddot{\xi} + \Lambda \xi = \tilde{V}^T \tilde{Q} \quad (3.3-53)$$

The coordinates  $\xi$  in this case are often referred to as vehicle normal modes.

The advantage of (3.3-53) over (3.3-50) from the algebraic point of view is apparent. Before we discuss state space form for the system equations we will translate the generalized forces  $\tilde{Q}$  into normalized control effort.

### 3.3.6 Normalized Control Effort

In chapter two it was pointed out that the control effort must be normalized. The control effort is imbedded in the generalized forces  $\tilde{Q}$  we derived in (3.3-39). The generalized forces are due to the control actuators. We will consider two types of actuators: 1) Force type, and 2) Torque type. The output of these actuators will be forces and pure couples, respectively. Any real actuator has a maximum output possible which we will term "actuator strength", and it is this maximum which

will be used to normalize the generalized forces. Referring to (3.3-39), let

$$\begin{aligned} Y_j &= Y_{j\max} u_j(t) \\ \tau_j &= \tau_{j\max} u_j(t) \end{aligned} \quad (3.3-54)$$

where  $Y_{j\max}$ ,  $\tau_{j\max}$  are the strength of the force actuator and the torque actuator, respectively, and  $j$  is any location for these actuators. The normalized control effort is  $u_j(t)$ . Note that  $u_j(t)$  is different for each type of actuator (i.e.,  $j$  belongs to different sets  $\mathcal{F}$ ,  $\mathcal{C}$ ). Substituting (3.3-54) in (3.3-39)

$$\begin{aligned} Q_0 &= \sum_{j \in \mathcal{F}} Y_{j\max} \tau'_{1j} u_j(t) + \sum_{j \in \mathcal{C}} \tau_{j\max} u_j(t) \\ Q_1 &= \sum_{j \in \mathcal{F}} \left[ \Phi^T(r_j) \delta_{Fj} - \frac{1}{m_s} \mu_\phi \right] Y_{j\max} u_j(t) \\ &\quad + \sum_{j \in \mathcal{C}} \Phi'^T(r_j) \delta_{Fj} \tau_{j\max} u_j(t) \end{aligned} \quad (3.3-55)$$

where  $\delta_{Fj}$  is zero for locations on  $B_R$  and unity for locations on  $B_F$ .

Define the following matrices:

$$H = \begin{bmatrix} H_0 \\ H_1 \end{bmatrix} ; \quad H_0 = [Y_{1\max} \tau'_{11}, Y_{2\max} \tau'_{12}, \dots, Y_{l\max} \tau'_{1l} ; \tau_{1\max}, \tau_{2\max}, \dots, \tau_{s\max}]$$

$$\{Y_{j\max} \tau'_{1j}\}, j \in \mathcal{F} ; \quad \{\tau_{j\max}\}, j \in \mathcal{C}$$

$$H_1 = [\hat{H}_{1\eta} \hat{H}_{2\eta} \dots \hat{H}_{l\eta} ; \hat{H}_{1\eta} \hat{H}_{2\eta} \dots \hat{H}_{s\eta}]$$

$$\{\hat{H}_{j\eta}\}, j \in \mathcal{F} ; \quad \{\hat{H}_{j\eta}\}, j \in \mathcal{C}$$

$$\hat{H}_{j\eta} = Y_{j\max} \left[ \Phi^T(r_j) \delta_{Fj} - \frac{1}{m_s} \mu_\phi \right], j = 1, 2, \dots, l \in \mathcal{F}$$

$$\hat{H}_{j\eta} = \tau_{j\max} \Phi'^T(r_j) \delta_{Fj}, j = 1, 2, \dots, s \in \mathcal{C}$$

$$u = \begin{bmatrix} u_F \\ u_c \end{bmatrix} ; \quad u_F^T = [u_1, u_2, \dots, u_l] ; \{u_i\}, i \in \mathcal{F} \\ u_c^T = [u_1, u_2, \dots, u_s] ; \{u_i\}, i \in \mathcal{C}$$

(3.3-56)

In this case  $H_0$  is a row vector and  $\hat{H}_{j_1}, \hat{H}_{j_2}$  are  $N \times 1$  columns. With these notations (3.3-55) can be written as:

$$\begin{aligned} Q_0 &= H_0 u \\ Q_1 &= H_1 u \end{aligned} \quad (3.3-57)$$

and from (3.3-43)

$$\hat{Q} = H u \quad (3.3-58)$$

The generalized forces are now in the form of normalized control effort  $u$ . Since we have the total number of controllers equal to  $n$ ,  $l + s = m$  in the above equations. The matrices  $H_0, H_1$  are of dimensions  $1 \times m$  and  $N \times m$ , respectively, so that  $H$  is of dimension  $(N+1) \times m$ . And  $u$  is of dimension  $m \times 1$ .

### 3.3.7 Normalized State Space Form for System Equations

We will consider the state space form for the equations of motion (3.3-50) and (3.3-53) in the two systems of coordinates  $\tilde{\eta}$  and  $\xi$ , respectively.

#### 1) $\tilde{\eta}$ system of coordinates

Defining

$$x^T = [\dot{e} \quad \dot{\eta}_1 \quad \dot{\eta}_2 \quad \dots \quad \dot{\eta}_n] \quad (3.3-59)$$

the equation

$$\ddot{\eta} + (\tilde{M}^{-1} \tilde{K}) \dot{\eta} = \tilde{M}^{-1} \tilde{Q} = \tilde{M}^{-1} H u$$

can be put in the form

$$\dot{x}^* = A^* x^* + B^* u \quad (3.3-60)$$

where  $A^*$  is an augmented matrix whose elements are elements of  $\tilde{M}^{-1} \tilde{K}$  and zeroes and unity at appropriate slots,  $B^*$  is an augmented matrix whose elements are elements of  $\tilde{M}^{-1} H$  and zeroes, such that (3.3-60) represents the system equations (3.3-50) and identity relations for all the elements of  $\dot{\eta}$ .

2)  $\xi$  system of coordinates

Define

$$x^T = [\xi_0 \quad \dot{\xi}_0 \quad \xi_1 \quad \dot{\xi}_1 \quad \xi_2 \quad \dot{\xi}_2 \quad \dots \quad \xi_n \quad \dot{\xi}_n] \quad (3.3-61)$$

Then the equation

$$\ddot{\xi} + \Lambda \xi = \tilde{V}^T \tilde{Q} = \tilde{V}^T H u$$

can be put in the form

$$\dot{x}^* = A^* x^* + B^* u \quad (3.3-62)$$

where  $A^*$ ,  $B^*$  are augmented matrices, similar to (1) above, with elements of  $\Lambda$  and  $\tilde{V}^T H$ , respectively, such that this equation represents the system equations (3.3-53) and identity relations for all the

elements of  $\dot{\xi}$ .

To obtain the final state space form corresponding to either of the two systems of coordinates,  $\tilde{\eta}$ ,  $\xi$ , normalization of the state variables  $x^*$  must be effected. This is done by assigning weighting factors to these variables so as to reflect the relative importance we attach in controlling them. Let  $N_i$ ,  $\bar{N}_i$ ,  $i=0,1,2,\dots,N$ , be a set of chosen weighting factors so that

$$x_{2i+1}^* = \frac{x_{2i+1}^*}{N_i} ; \quad x_{2i+2}^* = \frac{x_{2i+2}^*}{\bar{N}_i}, \quad i=0,1,2,\dots,N \quad (3.3-63)$$

where  $x_j^*$ ,  $x_j$ ,  $j=1,2,\dots,2N+2$ , are elements of  $x^*$ ,  $x$ , respectively.

For the  $\tilde{\eta}$  system

$$x_1^* = 0 ; \quad x_2^* = \dot{0} ; \quad x_{2i+1}^* = \dot{\eta}_i ; \quad x_{2i+2}^* = \ddot{\eta}_i \\ i = 1, 2, \dots, N \quad (3.3-64)$$

and for the  $\xi$  system

$$x_{2i+1}^* = \xi_i ; \quad x_{2i+2}^* = \dot{\xi}_i, \quad i = 0, 1, 2, \dots, N \quad (3.3-65)$$

From (3.3-53) we can write

$$x^* = D_N x \quad (3.3-66)$$

where

$$D_N = \text{diag} [ N_0 \quad \bar{N}_0 \quad N_1 \quad \bar{N}_1 \quad \dots \quad N_N \quad \bar{N}_N ] \quad (3.3-67)$$

Substituting for  $x^*$  from (3.3-55) in (3.3-50) or (3.3-52) and premultiplying by  $D_N^{-1}$  we obtain the final state space form

$$\dot{x} = Ax + Bu \quad (3.3-68)$$

where

$$A = D_N^{-1} A^* D_N \quad ; \quad B = D_N^{-1} B^* \quad (3.3-69)$$

Thus, the equations of motion in either of the two systems of coordinates  $\tilde{\eta}$  ,  $\xi$  can be put in the normalized state space form (3.3-63). It should, however, be noted that the factors  $N_i$  ,  $\bar{N}_i$  are not the same for both the systems  $\tilde{\eta}$  and  $\xi$  . The coordinates  $\tilde{\eta}$  are physically more meaningful than  $\xi$  which are abstract (a combination of hybrid coordinates  $\theta$  ,  $\eta$  ) and difficult to interpret in physical terms. As such it is easier to assign the weighting factors  $N_i$  ,  $\bar{N}_i$  for the  $\tilde{\eta}$  system than for the  $\xi$  system. Nevertheless, it may be possible to translate a set  $N_i$  ,  $\bar{N}_i$  in  $\tilde{\eta}$  system to an equivalent set  $N_i$  ,  $\bar{N}_i$  in  $\xi$  system using some understanding of the system behavior, etc.. However, in the present analysis we will assume that it is possible to obtain a set  $N_i$  ,  $\bar{N}_i$  in either system of coordinates. Also, note that any one of the  $N_i$  can be set equal to unity since the weighting factors reflect only the relative importance among the coordinates.

Before ending discussion on equations of motion we will derive expressions for the mode shape integrals based on this model.

### 3.3.3 Mode Shape Integrals

The mode shape integrals  $M_{\phi_i}$  ,  $M_{\tau\phi_i}$  ,  $\Delta_{ij}$  , and  $K_{ij}$  ,  $i=1,2,\dots,N$  ,  $j=1,2,\dots,N$  in (3.3-20) and (3.3-25) need to be evaluated to obtain the constants of the equations of motion. These integrals are taken over the flexible body  $B_F$  which is considered as one body, and hence any mode applies to the entire body. Because  $B_F$  is split about a rigid body these integrals are evaluated as explained in the following.

A mode shape is chosen separately for each appendage whose fixed base is the rigid body, and these two mode shapes for both the appendages together describe a mode shape for the flexible body  $B_F$ . Thus, there is a discontinuity in a mode shape of  $B_F$  due to the rigid body. Let  $\phi_j(r)$  denote a mode shape of the flexible body  $B_F$ , and define

$$\phi_j(r) = \begin{cases} \tilde{\phi}_j(x) & , \quad -(l^- + r_e^-) \leq r \leq -r_e^- \\ \hat{\phi}_j(x) & , \quad r_e^+ \leq r \leq l^+ + r_e^+ \end{cases} \quad (3.3-70)$$

where  $\tilde{\phi}_j(x)$  applies to the appendage in  $-\hat{b}_1$  direction (left) and  $\hat{\phi}_j(x)$  applies to the appendage in  $+\hat{b}_1$  direction (right);  $r$  is as defined before and  $x$  is measured from the rigid body base along  $\hat{b}_1$  direction;  $l^+$ ,  $l^-$  are the lengths of the appendages (from the rigid body base) on the positive (right) and negative (left) sides of the  $\hat{b}_1$  direction, and  $r_e^+$ ,  $r_e^-$  are the offsets of these appendages about the base of  $B_F$  in  $+\hat{b}_1$ ,  $-\hat{b}_1$  directions, respectively.

The mode shape integrals from (3.3-20) and (3.3-25) can now be expressed as:

$$\begin{aligned} \text{(i)} \quad \mu_{\phi_j} &= \int_{B_F} \phi_j(r) dm = \int_{B_F^-} \tilde{\phi}_j(x) dm + \int_{B_F^+} \hat{\phi}_j(x) dm \\ \text{(ii)} \quad \mu_{r\phi_j} &= \int_{B_F} r \phi_j(r) dm = \int_{B_F^-} r \tilde{\phi}_j(x) dm + \int_{B_F^+} r \hat{\phi}_j(x) dm \\ \text{(iii)} \quad \Delta_{jj} &= \int_{B_F} \phi_j(r) \phi_j(r) dm = \int_{B_F^-} \tilde{\phi}_j(x) \tilde{\phi}_j(x) dm + \int_{B_F^+} \hat{\phi}_j(x) \hat{\phi}_j(x) dm \end{aligned}$$

$$K_{ij} = \int_{B_F} (EI) \phi_i''(r) \phi_j''(r) dr \quad (3.3-71)$$

$$= \int_{B_F^-} (EI)^- \left[ \frac{d^2}{dr^2} \hat{\phi}_i(x) \right] \left[ \frac{d^2}{dr^2} \hat{\phi}_j(x) \right] dx \\ + \int_{B_F^+} (EI)^+ \left[ \frac{d^2}{dr^2} \hat{\phi}_i(x) \right] \left[ \frac{d^2}{dr^2} \hat{\phi}_j(x) \right] dx$$

where  $B_F^-$ ,  $B_F^+$  are used to denote the left and right appendages of  $B_F$ , respectively, and  $(EI)^-$ ,  $(EI)^+$  are used similarly. Let

$$dm = \begin{cases} c_\sigma^-(r) dr & \text{for } B_F^- \\ c_\sigma^+(r) dr & \text{for } B_F^+ \end{cases} \quad (3.3-72)$$

denote a differential mass, where  $c_\sigma^-(r)$ ,  $c_\sigma^+(r)$  are linear mass densities for  $B_F^-$ ,  $B_F^+$ , respectively. Note that  $r$  can be expressed in terms of  $x$  for  $B_F^-$ ,  $B_F^+$  as follows:

$$r = \begin{cases} x - r_e^- & , \quad -(l^- + r_e^-) \leq r \leq -r_e^- \\ x + r_e^+ & , \quad r_e^+ \leq r \leq l^+ + r_e^+ \end{cases} \quad (3.3-73)$$

from which

$$x = \begin{cases} r + r_e^- & , \quad -l^- \leq x \leq 0 & , \quad B_F^- \\ r - r_e^+ & , \quad 0 \leq x \leq l^+ & , \quad B_F^+ \end{cases} \quad (3.3-74)$$

With these we can write

$$\int_{B_F^-} (f^-) dm = \int_{-(l^- + r_e^-)}^{-r_e^-} (f^-) c_\sigma^-(r) dr = \int_{-l^-}^0 (f^-) c_\sigma^-(r) dx \\ \int_{B_F^+} (f^+) dm = \int_{r_e^+}^{l^+ + r_e^+} (f^+) c_\sigma^+(r) dr = \int_0^{l^+} (f^+) c_\sigma^+(r) dx \quad (3.3-75)$$



where  $f^-$ ,  $f^+$  are any functions corresponding to  $B_F^-$ ,  $B_F^+$ , respectively. The integration of the mode shapes can be put in a nondimensionalized form as follows.

Define

$$\gamma = \begin{cases} \gamma^* l^- & \text{for } B_F^- \\ \gamma^* l^+ & \text{for } B_F^+ \end{cases}; \quad x = \begin{cases} x^* l^- & \text{for } B_F^- \\ x^* l^+ & \text{for } B_F^+ \end{cases}$$

$$\phi_j(\gamma) = \epsilon_j \phi_j^*(\gamma^*) = \begin{cases} \epsilon_j \tilde{\phi}_j^*(x^*) & \text{for } B_F^- \\ \epsilon_j \hat{\phi}_j^*(x^*) & \text{for } B_F^+ \end{cases}$$

$$\gamma_e^- = (\gamma_e^-)^* l^- \quad \text{for } B_F^-$$

$$\gamma_e^+ = (\gamma_e^+)^* l^+ \quad \text{for } B_F^+$$

(3.3-76)

where  $\epsilon_j$  is some constant having a dimension of length and  $\tilde{\phi}_j^*$ ,  $\hat{\phi}_j^*$  are nondimensional functions. These lead to, from (3.3-73), (3.3-70) and (3.3-72),

$$\gamma^* = \begin{cases} x^* - (\gamma_e^-)^* & \text{for } B_F^- \\ x^* + (\gamma_e^+)^* & \text{for } B_F^+ \end{cases}; \quad dm = \begin{cases} c_F^-(\gamma) l^- d\gamma^* & \text{for } B_F^- \\ c_F^+(\gamma) l^+ d\gamma^* & \text{for } B_F^+ \end{cases}$$

$$\tilde{\phi}_j(x) = \epsilon_j \tilde{\phi}_j^*(x^*), \quad -1 \leq x^* \leq 0$$

$$\hat{\phi}_j(x) = \epsilon_j \hat{\phi}_j^*(x^*), \quad 0 \leq x^* \leq 1$$

$$\frac{d^2}{d\gamma^2} \tilde{\phi}_j(x) = \frac{d^2}{dx^2} \tilde{\phi}_j(x) = \frac{\epsilon_j}{(l^-)^2} \frac{d^2}{dx^{*2}} \tilde{\phi}_j^*(x^*)$$

$$\frac{d^2}{d\gamma^2} \hat{\phi}_j(x) = \frac{d^2}{dx^2} \hat{\phi}_j(x) = \frac{\epsilon_j}{(l^+)^2} \frac{d^2}{dx^{*2}} \hat{\phi}_j^*(x^*)$$

(3.3-77)

In nondimensional form (3.3-75) can be written as:

$$\int_{B_F^-} (f^-) dm = \int_{-l^-}^0 (f^-) c_\sigma^-(r) dx = \int_{-1}^0 (f^-) c_\sigma^-(r) l^- dx^*$$

$$\int_{B_F^+} (f^+) dm = \int_0^{l^+} (f^+) c_\sigma^+(r) dx = \int_0^1 (f^+) c_\sigma^+(r) l^+ dx^*$$

(3.3-78)

Using (3.3-75), (3.3-77), (3.3-73) and substituting for  $f^-$ ,  $f^+$  appropriately in (3.3-71) we obtain the following expressions for the mode shape integrals.

$$(i) \quad \mu_{\phi_j} = \int_{-1}^0 c_j^- \tilde{\phi}_j^*(x^*) dx^* + \int_0^1 c_j^+ \hat{\phi}_j^*(x^*) dx^*$$

$$(ii) \quad \mu_{r\phi_j} = \int_{-1}^0 c_j^- l^- x^* \tilde{\phi}_j^*(x^*) dx^* - \int_{-1}^0 c_j^- l^- (r_e^-)^* \tilde{\phi}_j^*(x^*) dx^*$$

$$+ \int_0^1 c_j^+ l^+ x^* \hat{\phi}_j^*(x^*) dx^* + \int_0^1 c_j^+ l^+ (r_e^+)^* \hat{\phi}_j^*(x^*) dx^*$$

$$(iii) \quad \Delta_{ij} = \int_{-1}^0 c_i^- \epsilon_i \tilde{\phi}_i^* \tilde{\phi}_j^* dx^* + \int_0^1 c_i^+ \epsilon_i \hat{\phi}_i^* \hat{\phi}_j^* dx^*$$

$$(iv) \quad k_{ij} = \int_{-1}^0 \frac{(\mathbb{E}I)^- \epsilon_i \epsilon_j}{(l^-)^3} \tilde{\phi}_i^{*''}(x^*) \tilde{\phi}_j^{*''}(x^*) dx^*$$

$$+ \int_0^1 \frac{(\mathbb{E}I)^+ \epsilon_i \epsilon_j}{(l^+)^3} \hat{\phi}_i^{*''}(x^*) \hat{\phi}_j^{*''}(x^*) dx^* \quad (3.3-79)$$

$$i = 1, 2, 3, \dots, N; \quad j = 1, 2, 3, \dots, N$$

where

$$\begin{aligned}
 c_j^- &= \epsilon_j c_j^-(r) l^- \\
 c_j^+ &= \epsilon_j c_j^+(r) l^+ \\
 \tilde{\phi}_j^{**} &= \frac{d^2}{dx^{*2}} \tilde{\phi}_j^* ; \quad \hat{\phi}_j^{**} = \frac{d^2}{dx^{*2}} \hat{\phi}_j^*(x^*)
 \end{aligned} \tag{3.3-90}$$

The expressions (3.3-79) will be used to evaluate the mode shape integrals for any chosen mode shapes. In the next chapter we will study a specific model of a large flexible spacecraft and obtain numerical results.

### 3.4 Summary

In this chapter we analyzed the dynamics of a typical model of a very large flexible spacecraft inertially at rest, and obtained equations of motion, linearized these equations about an equilibrium state and put them in state space form to be studied from the controllability point of view. The equations were derived for planar motion, and the flexible body was assumed to be elastic beams undergoing transverse deformation. Linearization was based on the assumption that the flexible body undergoes small deformations.

#### 4. APPLICATIONS TO A SPECIFIC MODEL OF A LARGE FLEXIBLE SPACECRAFT

In the last chapter a general model of a large flexible spacecraft was analyzed and equations of motion were derived. In this chapter we will choose a specific model in order to obtain numerical results and apply the concepts of the degree of controllability developed in chapter Two to study the effect of actuator locations on the controllability of the system.

#### 4.1 Model Description

Figure 5 shows a specific model of a large flexible spacecraft. It consists of a cylindrical rigid body and two identical appendages, i.e., the length, linear mass density and stiffness of both the appendages are identical, the latter two properties being uniform throughout the length of these appendages. The offsets of these appendages about the  $\hat{b}_2$  axis (in the rigid body) are equal. Each appendage is treated individually as a cantilever beam with its root in the rigid body as the fixed end. Any cantilever mode considered for the two appendages taken together constitutes a single mode for the entire flexible body  $B_F$ . For any cantilever mode of an appendage two types of modes for the flexible body  $B_F$  will be considered:

- 1) Symmetric mode -- in which a cantilever mode of an appendage is imposed symmetrically about the rigid body for the two appendages, i.e., the appendages execute a symmetric mode of motion as in Fig. 6.
- 2) Antisymmetric mode -- in which a cantilever mode of an appendage is imposed antisymmetrically about the rigid body for the two appendages, i.e., the appendages execute an antisymmetric mode of motion as shown in Fig. 7. In this analysis we will treat separately the symmetric and antisymmetric modes of motion of the flexible body  $B_F$ .

## 4.2 Degree of Controllability

### 4.2.1 Mode Shape Integrals

For the model described above, we can write, in the notations of section 3.3.8,

$$l^- = l^+ = l \quad ; \quad \gamma_e^- = \gamma_e^+ = \gamma_e \quad ; \quad (\gamma_e^-)^* = (\gamma_e^+)^* = \gamma_e^* \\ c_r^-(\gamma) = c_r^+(\gamma) = c_r = \text{constant} \quad ; \quad (EI)^- = (EI)^+ = (EI) = \text{constant}$$

$$m_F = 2 c_r l$$

$$\phi_j(\gamma) = \begin{cases} \tilde{\phi}_j(x) & , \quad -(l + \gamma_e) \leq \gamma \leq -\gamma_e \\ \hat{\phi}_j(x) & \gamma_e \leq \gamma \leq l + \gamma_e \end{cases}$$

$$\tilde{\phi}_j(x) = \epsilon_j \tilde{\phi}_j^*(x^*) \quad , \quad -1 \leq x^* \leq 0$$

$$\hat{\phi}_j(x) = \epsilon_j \hat{\phi}_j^*(x^*) \quad , \quad 0 \leq x^* \leq 1$$

$$c_j^- = c_j^+ = c_j = \epsilon_j c_r l = \text{constant} \quad ; \quad \epsilon_j = \frac{2 c_j}{m_F}$$

$$j = 1, 2, \dots, N$$

(4.2-1)

where  $\tilde{\phi}_j, \hat{\phi}_j$  apply to the left and right appendages, respectively, and the asterisked quantities are nondimensional. Using these in (3.3-78) the expressions for the mode shape integrals become

$$(i) \mu_{\phi_j} = c_j \left[ \int_{-1}^0 \tilde{\phi}_j^*(x^*) dx^* + \int_0^1 \hat{\phi}_j^*(x^*) dx^* \right]$$

$$(ii) \mu_{r\phi_j} = c_j l \left[ \int_{-1}^0 x^* \tilde{\phi}_j^*(x^*) dx^* + \int_0^1 x^* \hat{\phi}_j^*(x^*) dx^* \right] \\ - c_j l r_0^* \left[ \int_{-1}^0 \tilde{\phi}_j^*(x^*) dx^* - \int_0^1 \hat{\phi}_j^*(x^*) dx^* \right]$$

$$(iii) \Delta_{ij} = \epsilon_i c_j \left[ \int_{-1}^0 \tilde{\phi}_i^* \tilde{\phi}_j^* dx^* + \int_0^1 \hat{\phi}_i^* \hat{\phi}_j^* dx^* \right]$$

$$(iv) k_{ij} = \frac{(EI) \epsilon_i \epsilon_j}{l^3} \left[ \int_{-1}^0 \tilde{\phi}_i^{*''}(x^*) \tilde{\phi}_j^{*''}(x^*) dx^* \right. \\ \left. + \int_0^1 \hat{\phi}_i^{*''}(x^*) \hat{\phi}_j^{*''}(x^*) dx^* \right]$$

$$i, j = 1, 2, \dots, N$$

(4.2-2)

The cantilever mode shapes [8] for an appendage can be written in nondimensional form as

$$\hat{\phi}_j^*(x^*) = (\cosh \beta_j l x^* - \cos \beta_j l x^*) - \gamma_j (\sinh \beta_j l x^* - \sin \beta_j l x^*)$$

$$0 \leq x^* \leq 1$$

$$j = 1, 2, \dots, N$$

(4.2-3)

where  $\beta_j$ ,  $\gamma_j$  are constants, and the mode shapes are written for the right appendage ( $B_F^+$ ). The modes for the entire flexible body  $B_F$  are obtained by choosing

$$\tilde{\phi}_j^*(x^*) = \delta \hat{\phi}_j^*(-x^*) \quad , \quad -1 \leq x^* \leq 0$$

$$\delta = \begin{cases} 1 & \text{for symmetric mode of } B_F \\ -1 & \text{for antisymmetric mode of } B_F \end{cases}$$

(4.2-4)



The mode shapes  $\tilde{\phi}_j^*$ ,  $\hat{\phi}_j^*$  together describe the  $j^{\text{th}}$  mode for  $B_F$ . Making a change of variable,  $x^* = -z^*$ , in the integrals involving  $\tilde{\phi}_j^*(x^*)$  in (4.2-2) these integrals can be expressed in terms of the integrals involving  $\hat{\phi}_j^*(x^*)$ . And the mode shape integrals in (4.2-2) can be expressed as:

$$\begin{aligned} \text{(i)} \quad M_{\phi_j} &= c_j (1+\delta) \int_0^1 \hat{\phi}_j^*(x^*) dx^* \\ \text{(ii)} \quad M_{\psi\phi_j} &= c_j l (1-\delta) \left[ \int_0^1 x^* \hat{\phi}_j^*(x^*) dx^* + \tau_e^* \int_0^1 \hat{\phi}_j^*(x^*) dx^* \right] \\ \text{(iii)} \quad \Delta_{ij} &= \epsilon_i c_j (1+\delta^2) \int_0^1 \hat{\phi}_i^*(x^*) \hat{\phi}_j^*(x^*) dx^* \\ \text{(iv)} \quad K_{ij} &= \frac{(EI) \epsilon_i \epsilon_j}{l^3} (1+\delta^2) \int_0^1 \hat{\phi}_i^{*''}(x^*) \hat{\phi}_j^{*''}(x^*) dx^* \end{aligned}$$

$i, j = 1, 2, \dots, N$

(4.2-5)

The cantilever mode shapes  $\hat{\phi}_j(x)$  (or  $\tilde{\phi}_j(x)$ ) are orthogonal functions with respect to the mass density  $c_r$ , and hence the following properties (see [6] for details) can be obtained:

$$\begin{aligned} \int_0^l \hat{\phi}_i(x) \hat{\phi}_j(x) c_r dx &= \alpha_j \delta_{ij} \\ \int_0^l (EI) \hat{\phi}_i^{*''}(x) \hat{\phi}_j^{*''}(x) dx &= \omega_j^2 \alpha_j \delta_{ij} \\ \omega_j^2 &= \frac{EI}{c_r} \beta_j^4 \quad ; \quad \hat{\phi}_i^{*''} = \frac{d^2}{dx^2} \hat{\phi}_i(x), \quad i, j = 1, 2, \dots, N \end{aligned} \quad (4.2-6)$$

where  $\alpha_j$  is a normalizing constant to be chosen later,  $\omega_j$  is the frequency for the  $j^{\text{th}}$  cantilever mode, and  $\delta_{ij}$  is the Kronecker delta.

These properties are true for  $\tilde{\phi}_j(x)$  also. In nondimensional form (4.2-6) yields

$$\begin{aligned} c_j \epsilon_i \int_0^1 \hat{\phi}_i^*(x^*) \hat{\phi}_j^*(x^*) dx^* &= \alpha_j \delta_{ij} \\ \frac{(EI) \epsilon_i \epsilon_j}{l^3} \int_0^1 \hat{\phi}_i^{*''}(x^*) \hat{\phi}_j^{*''}(x^*) dx^* &= \omega_j^2 \alpha_j \delta_{ij} \end{aligned} \quad (4.2-7)$$

Now, define

$$\begin{aligned} a_j &= \int_0^1 \hat{\phi}_j^*(x^*) dx^* \\ e_j &= \int_0^1 x^* \hat{\phi}_j^*(x^*) dx^* \\ g_j &= \int_0^1 [\hat{\phi}_j^*(x^*)]^2 dx^* \end{aligned} \quad (4.2-8)$$

which by the substitution of (4.2-3), can be obtained from the following:

$$\begin{aligned} a_j &= \frac{1}{\beta_j l} \left[ 2\gamma_j + (\sinh \beta_j l - \sin \beta_j l) - \gamma_j (\cosh \beta_j l + \cos \beta_j l) \right] \\ e_j &= \frac{1}{\beta_j l} \left\{ (\sinh \beta_j l - \sin \beta_j l) - \gamma_j (\cosh \beta_j l + \cos \beta_j l) \right. \\ &\quad \left. + \frac{1}{\beta_j l} \left[ 2 + \gamma_j (\sinh \beta_j l + \sin \beta_j l) - (\cosh \beta_j l + \cos \beta_j l) \right] \right\} \\ g_j &= 1 + \frac{1}{\beta_j l} \left[ 2\gamma_j \sin \beta_j l \sinh \beta_j l - (1 + \gamma_j^2) \sin \beta_j l \cosh \beta_j l \right. \\ &\quad \left. - (1 - \gamma_j^2) \cos \beta_j l \sinh \beta_j l \right] + \frac{1}{2\beta_j l} \left[ \left( \frac{1 - \gamma_j^2}{\gamma_j} \right) \sin 2\beta_j l \right. \\ &\quad \left. + \left( \frac{1 + \gamma_j^2}{2} \right) \sinh 2\beta_j l + \gamma_j (\cos 2\beta_j l - \cosh 2\beta_j l) \right] \end{aligned}$$

(4.2-9)

Using (4.2-7) and (4.2-8) in (4.2-5) the mode shape integrals can be expressed as:

$$\begin{aligned}
 (i) \quad \mu_{\phi_j} &= c_j (1 + \delta) a_j \\
 (ii) \quad \mu_{\tau\phi_j} &= c_j l (1 - \delta) (e_j + \tau e^* a_j) \\
 (iii) \quad \Delta_{jj} &= 2 c_j \epsilon_j g_j \delta_{jj} \\
 (iv) \quad k_{jj} &= 2 \omega_j^2 c_j \epsilon_j g_j \delta_{jj}
 \end{aligned} \tag{4.2-10}$$

where  $\alpha_j = c_j \epsilon_j g_j$  and  $\delta^2 = 1$  have been substituted.

Although one could consider many mode shapes we will restrict here to two ( $j=1,2$ ) cantilever mode shapes of an appendage, the first and the second. This gives four modes for  $B_F$ , two symmetric and two antisymmetric, which are treated separately. Equations (4.2-10) represent two sets of mode shape integrals for  $B_F$ , one for the symmetric mode ( $\delta = 1$ ) and the other for the antisymmetric mode ( $\delta = -1$ ).

Table 1 shows the various constants related to the first and second cantilever modes given by (4.2-3) for  $j = 1, 2$ . The nodes for a  $j^{\text{th}}$  mode are denoted by  $x_{ej}^*$  in the table.

#### 4.2.2 Computation of Inertia and Stiffness Matrices

From (3.3-22),

$$\begin{aligned}
 M &= \Delta - \frac{1}{m_s} \mu_{\phi} \mu_{\phi}^T \\
 \mu &= \mu_{\tau} - \frac{c_{g2}}{m_s} \mu_{\phi}
 \end{aligned} \tag{4.2-11}$$

if  $M_{ij}$ ,  $\mu_{ij}$  denote the elements of  $M, \mu$ , respectively, then substituting for  $\Delta_{ij}$ ,  $\mu_{ij}$ ,  $\mu_{ij}$  from (4.2-10), and  $e_j$  from (4.2-1) we obtain

$$\begin{aligned} M_{ij} &= \frac{4C_j^2}{m_F} g_j \delta_{ij} - \frac{1}{m_S} C_i C_j (1+\delta)^2 a_i a_j \\ \mu_j &= C_j l \left[ (1-\delta)(e_j + \tau e^* a_j) - \frac{C_{j2}}{m_j l} (1+\delta) a_j \right] \\ k_{ij} &= \frac{4C_j^2}{m_F} \omega_j^2 g_j \delta_{ij} \end{aligned} \quad (4.2-12)$$

We will now use the freedom to choose the normalizing constant  $C_j$  (see (4.2-5)) so as to obtain some algebraic advantage. If we let

$$M_{jj} = I_s, \quad j = 1, 2 \quad (4.2-13)$$

then

$$\begin{aligned} C_j &= \frac{\sqrt{m_S I_s}}{D_j} \\ D_j &= \sqrt{4 \frac{m_S}{m_F} g_j - (1+\delta)^2 a_j^2} \\ j &= 1, 2 \end{aligned} \quad (4.2-14)$$

Substituting for  $C_j$  from this in (4.2-12)

$$\begin{aligned} M_{ij} &= \begin{cases} I_s, & i=j \\ -I_s \frac{(1+\delta)^2 a_i a_j}{D_i D_j}, & i \neq j \end{cases} \\ \mu_j &= I_s \sqrt{\frac{m_S l^2}{I_s}} \frac{1}{D_j} \left[ (1-\delta)(e_j + \tau e^* a_j) - \left( \frac{C_{j2}}{m_j l} \right) (1+\delta) a_j \right] \\ k_{ij} &= I_s 4 \frac{m_S}{m_F} g_j \frac{\omega_j^2}{D_j^2} \delta_{ij} \end{aligned} \quad (4.2-15)$$

The data for the model of Fig. 2 are given in Table 2, and Table 3 shows the required input data derived based on the assumptions for this model. Table 4 shows the frequencies  $\omega_j$  of the cantilever modes and the constants  $D_j$  for both symmetric and antisymmetric modes of  $B_F$ . Using Tables 1, 3 and 4 the elements  $M_{ij}$ ,  $\mu_j$ ,  $K_{ij}$  in (4.3-5) are evaluated for symmetric ( $\delta = 1$ ) and antisymmetric ( $\delta = -1$ ) modes of  $B_F$  and result in the following matrices  $M$ ,  $\mu$ ,  $K$ ,  $\tilde{M}$ ,  $\tilde{K}$ :

1) Symmetric mode of motion of  $B_F$  ( $\delta = 1$ )

$$M = I_3 \begin{bmatrix} 1.000 & -3.12411 \times 10^{-4} \\ -3.12411 \times 10^{-4} & 1.000 \end{bmatrix}$$

$$\mu^T = I_3 \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$K = I_3 \begin{bmatrix} 3.294051 \times 10^{-2} & 0 \\ 0 & 1.293384 \end{bmatrix}$$

$$\tilde{M} = I_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3.12411 \times 10^{-4} \\ 0 & -3.12411 \times 10^{-4} & 1 \end{bmatrix}$$

$$\tilde{K} = I_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3.294051 \times 10^{-2} & 0 \\ 0 & 0 & 1.293384 \end{bmatrix}$$

(4.2-16)

2) Antisymmetric mode of motion of  $B_F$  ( $\delta = -1$ )

$$M = I_s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \mu^T = I_s [0.603751, 0.1065026]$$

$$K = I_s \begin{bmatrix} 3.292193 \times 10^{-2} & 0 \\ 0 & 1.293160 \end{bmatrix}$$

$$\tilde{M} = I_s \begin{bmatrix} 1 & 0.603751 & 0.1065026 \\ 0.603751 & 1 & 0 \\ 0.1065026 & 0 & 1 \end{bmatrix}$$

$$\tilde{K} = I_s \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3.292193 \times 10^{-2} & 0 \\ 0 & 0 & 1.293160 \end{bmatrix}$$

(4.2-17)

where  $\tilde{M}$ ,  $\tilde{K}$  are the inertia and the stiffness matrices and are defined in (3.3-48).

#### 4.2.3 Computation of Matrices for Generalized Forces

From (3.3-55)

$$H = \begin{bmatrix} H_0 \\ H_\eta \end{bmatrix}$$

$$H_0 = [Y_{1max} \gamma'_{11}, Y_{2max} \gamma'_{12}, \dots, Y_{lmax} \gamma'_{1l} | \tau_{1max}, \tau_{2max}, \dots, \tau_{5max}]$$

$$H_\eta = [\hat{H}_{1\eta} \hat{H}_{2\eta} \dots \hat{H}_{L\eta} ; \hat{H}_{1\eta} \hat{H}_{2\eta} \dots \hat{H}_{s\eta}]$$

$$\hat{H}_{i\eta} = \gamma_{i\max} [\Phi^T(\gamma_i) \delta_{Fi} - \frac{1}{m_s} \mu_p] , \quad i \in J$$

$$\hat{H}_{i\eta} = \tau_{i\max} \Phi'^T(\gamma_i) \delta_{Fi} , \quad i \in C$$

(4.2-18)

in which  $i$  refers to the actuators and  $\delta_{Fi}$  is defined in (3.3-40).

Also, define

$$k_{Fi} = \frac{\gamma_{i\max} l}{I_s}$$

$$k_{ci} = \frac{\tau_{i\max}}{I_s}$$

$$\gamma_{ii}^* = \frac{\gamma_{ii}'}{l}$$

(4.2-19)

where  $k_{Fi}$ ,  $k_{ci}$  are some constants related to the actuator strengths.

Now,

$$\Phi^T(\gamma_i) = \begin{bmatrix} \phi_1(\gamma_i) \\ \phi_2(\gamma_i) \end{bmatrix} = \begin{bmatrix} \epsilon_1 \phi_1^*(\gamma_i^*) \\ \epsilon_2 \phi_2^*(\gamma_i^*) \end{bmatrix}$$

$$\Phi'^T(\gamma_i) = \begin{bmatrix} \phi_1'(\gamma_i) \\ \phi_2'(\gamma_i) \end{bmatrix} = \begin{bmatrix} \frac{\epsilon_1}{l} \phi_1^{*'}(\gamma_i^*) \\ \frac{\epsilon_2}{l} \phi_2^{*'}(\gamma_i^*) \end{bmatrix}$$

(4.2-20)

where

$$\phi_i^*(\gamma_i^*) = \begin{cases} \hat{\phi}_i^*(x_i^*) & , \quad -1 \leq x_i^* \leq 0 \\ \hat{\hat{\phi}}_i^*(x_i^*) & , \quad 0 \leq x_i^* \leq 1 \end{cases}$$

$$\hat{\phi}_j^*(x_i^*) = \delta \hat{\phi}_j^*(-x_i^*)$$

$$\delta = \begin{cases} 1 & \text{for symmetric mode of } \theta_F \\ -1 & \text{for antisymmetric mode of } \theta_F \end{cases}$$

$$\begin{aligned} \phi_j^{*'}(\tau_i^*) &= \frac{d}{d\tau^*} \phi_j^*(\tau^*) \Big|_{\tau^* = \tau_i^*} \\ &= \begin{cases} \hat{\phi}_j^{*'}(x_i^*) & , \quad -1 \leq x_i^* \leq 0 \\ \hat{\phi}_j^{*'}(x_i^*) & , \quad 0 \leq x_i^* \leq 1 \end{cases} \end{aligned}$$

$$\tilde{\phi}_j^{*'}(x_i^*) = -\delta \hat{\phi}_j^{*'}(x_i^*) , \quad 0 \leq x_i^* \leq 1 \quad (4.2-21)$$

Substituting (4.2-19) and (4.2-20) in (4.2-18) and using (4.2-10), (4.2-1) and (4.2-14) for  $\mu_{\phi_j}$ ,  $\epsilon_j$ ,  $C_j$ , respectively, we obtain:

$$H_0 = I_s \left[ k_{F1} \gamma_{11}^{'*}, k_{F2} \gamma_{12}^{'*}, \dots, k_{F\ell} \gamma_{1\ell}^{'*} ; k_{c1}, k_{c2}, \dots, k_{cs} \right]$$

$$\hat{H}_{i\eta} = I_s \sqrt{\frac{I_s}{m_s l^2}} k_{Fi} \begin{bmatrix} \frac{1}{D_1} 2 \frac{m_s}{m_F} \phi_1^*(\tau_i^*) \delta_{Fi} - (1+\delta) a_1 \\ \frac{1}{D_2} 2 \frac{m_s}{m_F} \phi_2^*(\tau_i^*) \delta_{Fi} - (1+\delta) a_2 \end{bmatrix}$$

$$\hat{H}_{ci} = I_s \left( 2 \frac{m_s}{m_F} \sqrt{\frac{I_s}{m_s l^2}} \right) k_{ci} \begin{bmatrix} \frac{1}{D_1} \phi_1^{*'}(\tau_i^*) \delta_{Fi} \\ \frac{1}{D_2} \phi_2^{*'}(\tau_i^*) \delta_{Fi} \end{bmatrix}$$

$$H_\eta = \left[ \hat{H}_{1\eta}, \hat{H}_{2\eta}, \dots, \hat{H}_{\ell\eta} ; \hat{H}_{1\eta}, \hat{H}_{2\eta}, \dots, \hat{H}_{s\eta} \right] \quad (4.2-22)$$

where the values of  $\frac{m_s}{m_F}$ ,  $\sqrt{\frac{I_s}{m_s l^2}}$ ,  $D_1$ ,  $D_2$  for our model are given in Tables 3 and 4. The distances  $r_{1i}^{'*}$ ,  $r_i^*$  represent the variable locations of the actuators in  $B_R$ ,  $B_F$ , respectively, measured in  $\hat{b}_1$  direction. Note that  $r_{1i}^{'*}$ ,  $r_i^*$  can be considered as one variable and



the  $\delta_{Fi}$  factor takes care of the situation of whether the actuator is on  $B_R$  or  $B_F$ . Substituting the values from Tables 3 and 4, (4.2-22) can be written as:

1) Symmetric mode of  $B_F$  ( $\delta = 1$ )

$$\hat{H}_{i\eta} = I_3 k_{Fi} W_i ; \quad \hat{H}_{ci\eta} = I_3 k_{ci} Z_i$$

$$W_i = \begin{bmatrix} w_1^{(i)} \\ w_2^{(i)} \end{bmatrix} = \begin{bmatrix} 0.98565902 \phi_1^A(\gamma_i^A) \delta_{Fi} - 0.00071014 \\ 0.98530145 \phi_2^A(\gamma_i^A) \delta_{Fi} - 0.00039314 \end{bmatrix}$$

$$Z_i = \begin{bmatrix} z_1^{(i)} \\ z_2^{(i)} \end{bmatrix} = \begin{bmatrix} 0.98565902 \phi_1^{A'}(\gamma_i^A) \delta_{Fi} \\ 0.98530145 \phi_2^{A'}(\gamma_i^A) \delta_{Fi} \end{bmatrix} \quad (4.2-23)$$

$$H = I_3 \left[ \begin{array}{ccc|cc} k_{F1} \gamma_{11}^{A'} & k_{F2} \gamma_{12}^{A'} & \dots & k_{F\ell} \gamma_{1\ell}^{A'} & k_{c1} & k_{c2} & \dots & k_{cs} \\ k_{F1} w_1^{(u)} & k_{F2} w_1^{(u)} & \dots & k_{F\ell} w_1^{(u)} & k_{c1} z_1^{(u)} & k_{c2} z_1^{(u)} & \dots & k_{cs} z_1^{(u)} \\ k_{F1} w_2^{(u)} & k_{F2} w_2^{(u)} & \dots & k_{F\ell} w_2^{(u)} & k_{c1} z_2^{(u)} & k_{c2} z_2^{(u)} & \dots & k_{cs} z_2^{(u)} \end{array} \right]$$

$$\tilde{Q} = HU$$

2) Antisymmetric mode of  $B_F$  ( $\delta = -1$ )

$$\hat{H}_{i\eta} = I_3 k_{Fi} W_i ; \quad \hat{H}_{ci\eta} = I_3 k_{ci} Z_i$$

$$W_i = \begin{bmatrix} w_1^{(i)} \\ w_2^{(i)} \end{bmatrix} = \begin{bmatrix} 0.98538098 \phi_1^A(\gamma_i^A) \delta_{Fi} \\ 0.98521622 \phi_2^A(\gamma_i^A) \delta_{Fi} \end{bmatrix}$$

$$Z_i = \begin{bmatrix} z_1^{(i)} \\ z_2^{(i)} \end{bmatrix} = \begin{bmatrix} 0.98538098 \phi_1^{A'}(\gamma_i^A) \delta_{Fi} \\ 0.98521622 \phi_2^{A'}(\gamma_i^A) \delta_{Fi} \end{bmatrix}$$

$$H = I_3 \left[ \begin{array}{ccc|cc} k_{F1} \gamma_{11}^{A'} & k_{F2} \gamma_{12}^{A'} & \dots & k_{F\ell} \gamma_{1\ell}^{A'} & k_{c1} & k_{c2} & \dots & k_{cs} \\ k_{F1} w_1^{(u)} & k_{F2} w_1^{(u)} & \dots & k_{F\ell} w_1^{(u)} & k_{c1} z_1^{(u)} & k_{c2} z_1^{(u)} & \dots & k_{cs} z_1^{(u)} \\ k_{F1} w_2^{(u)} & k_{F2} w_2^{(u)} & \dots & k_{F\ell} w_2^{(u)} & k_{c1} z_2^{(u)} & k_{c2} z_2^{(u)} & \dots & k_{cs} z_2^{(u)} \end{array} \right]$$

$$\tilde{Q} = HU \quad (4.2-24)$$

where  $\phi_j^*(r_i^*)$ ,  $\phi_j^{*'}(r_i^*)$  are to be appropriately substituted according to (4.2-21), and  $\bar{Q}$  is defined in (3.3-48). The vectors  $W_i$ ,  $\bar{Z}_i$  are completely determined for any  $r_i^*$ .

#### 4.2.4 State Space Form for System Equations

Having obtained the necessary matrices we can now write the equations of motion (3.3-49) for symmetric and antisymmetric modes of  $B_F$  as

$$\bar{M} \ddot{\bar{\eta}} + \bar{K} \bar{\eta} = \bar{Q} \quad ; \quad \bar{\eta} = \begin{bmatrix} \theta \\ \eta \end{bmatrix} \quad (4.2-25)$$

where  $\bar{M}$ ,  $\bar{K}$ ,  $\bar{Q}$  are obtained from (4.2-16) and (4.2-23) for the symmetric mode of  $B_F$ , and from (4.2-17) and (4.2-24) for the antisymmetric mode of  $B_F$ . To obtain the orthogonal form of these equations (vehicle modes) we diagonalize  $\bar{M}$ ,  $\bar{K}$  by the principal matrix  $\hat{V}$  as explained in section 3.3.5. The principal matrix is given by

$$\hat{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for symmetric mode of } B_F$$

$$\hat{V} = \begin{bmatrix} 1 & 0.756797 & 0.171629 \\ 0 & -1.254086 & -0.106278 \\ 0 & 0.001363 & -1.009040 \end{bmatrix} \quad \begin{array}{l} \text{for antisymmetric} \\ \text{mode of } B_F \end{array}$$

(4.2-26)

The principal matrix for the symmetric mode of  $B_F$  is taken as the unit matrix within the working accuracy of the numerical terms. This means  $\tilde{M}$  in (4.2-16) for symmetric mode is approximated as a diagonal matrix (the off diagonal elements are much smaller than the diagonal elements). Or,

$$\tilde{M} = I_s E \quad \text{for symmetric mode of } B_F \quad (4.2-27)$$

where  $E$  is the unit matrix. The orthogonal form of the equations, from (3.3-53), is

$$\ddot{\xi} + \Lambda \xi = \tilde{V}^T \frac{\tilde{Q}}{I_s} \quad (4.2-28)$$

where

$$\begin{aligned} \xi^T &= [\xi_0 \quad \xi_1 \quad \xi_2 \quad \dots \quad \xi_n] \\ \Lambda &= \frac{1}{I_s} \tilde{V}^T K \tilde{V} = \text{diag} [\lambda_1 \quad \lambda_2 \quad \lambda_3] \end{aligned} \quad (4.2-29)$$

The coordinates  $\xi$  are vehicle normal modes. For the symmetric mode of  $B_F$  these are the same as  $\tilde{\eta}$  coordinates. Table 5 gives the values of  $\lambda_i$  for both symmetric and antisymmetric modes of  $B_F$ . Now, let

$$\frac{1}{I_s} H = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1m} \\ h_{21} & h_{22} & \dots & h_{2m} \\ h_{31} & h_{32} & \dots & h_{3m} \end{bmatrix} \quad (4.2-30)$$

where  $H$  is given by (4.2-23) or (4.2-24) for symmetric or antisymmetric mode of  $B_F$ . Then

$$\begin{aligned} \tilde{V}^T \frac{\hat{Q}}{I_3} &= \tilde{V}^T \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1m} \\ h_{21} & h_{22} & \dots & h_{2m} \\ h_{31} & h_{32} & \dots & h_{3m} \end{bmatrix} u \\ &= \begin{bmatrix} \hat{h}_{11} & \hat{h}_{12} & \dots & \hat{h}_{1m} \\ \hat{h}_{21} & \hat{h}_{22} & \dots & \hat{h}_{2m} \\ \hat{h}_{31} & \hat{h}_{32} & \dots & \hat{h}_{3m} \end{bmatrix} \end{aligned}$$

(4.2-31)

where

$$\begin{aligned} \hat{h}_{ki} &= h_{ki} \quad \text{for symmetric mode of } B_F \\ \hat{h}_{ki} &= \sum_{j=1}^3 \tilde{v}_{jk} h_{ji} \quad \text{for antisymmetric mode of } B_F \end{aligned} \quad (4.2-32)$$

and  $\tilde{v}_{jk}$  are elements of  $\tilde{V}$ ,  $j, k=1, 2, 3$ ,  $i=1, 2, \dots, m$ . From (4.2-32), (4.2-23), (4.2-24)

$$\hat{h}_{ki} = \tilde{v}_{1k} h_{1i} + \tilde{v}_{2k} h_{2i} + \tilde{v}_{3k} h_{3i}, \quad k=1, 2, 3$$

$$\left. \begin{aligned} h_{1i} &= k_{Fi} \gamma_{1i}' \\ h_{2i} &= k_{Fi} w_1^{(i)} \\ h_{3i} &= k_{Fi} w_2^{(i)} \end{aligned} \right\} \text{ or } \begin{aligned} h_{1i} &= k_{ci} \\ h_{2i} &= k_{ci} z_1^{(i)} \\ h_{3i} &= k_{ci} z_2^{(i)} \end{aligned}$$

(4.2-33)

where for symmetric modes of  $B_F$   $\tilde{v}_{jk} = \delta_{jk}$ , and for  $h_{ki}$ ,  $k=1, 2, 3$ , the first or the second set can be used for any  $i$  ( $1, 2, \dots, m$ ) depending on

the type (force or torque) of an actuator. Of course, appropriate vectors  $W_i$ ,  $Z_i$  must be used for symmetric or antisymmetric modes of  $B_F$ . Substituting for  $h_{ki}$  from these we obtain:

$$\hat{h}_{1i} = k_{Fi} (\tilde{v}_{11} \gamma'_{1i} + \tilde{v}_{21} w_1^{(i)} + \tilde{v}_{31} w_2^{(i)})$$

$$\hat{h}_{2i} = k_{Fi} (\tilde{v}_{12} \gamma'_{1i} + \tilde{v}_{22} w_1^{(i)} + \tilde{v}_{32} w_2^{(i)})$$

$$\hat{h}_{3i} = k_{Fi} (\tilde{v}_{13} \gamma'_{1i} + \tilde{v}_{23} w_1^{(i)} + \tilde{v}_{33} w_2^{(i)})$$

for force type of actuators for any  $i$ ,  $i=1,2,\dots,m$

$$\hat{h}_{1i} = k_{ci} (\tilde{v}_{11} + \tilde{v}_{21} z_1^{(i)} + \tilde{v}_{31} z_2^{(i)})$$

$$\hat{h}_{2i} = k_{ci} (\tilde{v}_{12} + \tilde{v}_{22} z_1^{(i)} + \tilde{v}_{32} z_2^{(i)})$$

$$\hat{h}_{3i} = k_{ci} (\tilde{v}_{13} + \tilde{v}_{23} z_1^{(i)} + \tilde{v}_{33} z_2^{(i)})$$

(4.2-34)

for torque type of actuators for any  $i$ ,  $i=1,2,\dots,m$

where  $\tilde{v}$ ,  $W_i$ ,  $Z_i$  are appropriately substituted from (4.2-26), (4.2-23), (4.2-24) for symmetric or antisymmetric mode of motion of  $B_F$ . Also, note that a combination of force and torque types of actuators can be used (substitute appropriate set for that  $i^{\text{th}}$  number; the total number of actuators is  $m$ ).

For the symmetric case these reduce to:

$$\hat{h}_{1i} = k_{Fi} \gamma'_{1i}$$

$$\hat{h}_{2i} = k_{Fi} (0.98565902 \phi_1^*(\gamma_i^*) \delta_{Fi} - 0.00071014)$$

$$\hat{h}_{3i} = k_{Fi} (0.98530145 \phi_2^*(\gamma_i^*) \delta_{Fi} - 0.00039314)$$

for force type actuators for any  $i$ ,  $i=1,2,\dots,m$ .

$$\hat{h}_{1i} = k_{ci}$$

$$\hat{h}_{2i} = k_{ci} (0.98565902) \phi_1^* (\tau_i^*) \delta_{Fi}$$

$$\hat{h}_{3i} = k_{ci} (0.98530145) \phi_2^* (\tau_i^*) \delta_{Fi}$$

for torque type actuators for any  $i, i=1,2,\dots,m$  (4.2-35)

The state space form of the equations (4.2-28) can now be obtained in the manner described in (3.3-61), (3.3-52), and for the system of vehicle normal modes  $\xi$  this is given by

$$\dot{X}^* = A^* X^* + B^* u \quad (4.2-36)$$

where

$$\begin{aligned} X^{*T} &= [\xi_0 \quad \dot{\xi}_0 \quad \xi_1 \quad \dot{\xi}_1 \quad \xi_2 \quad \dot{\xi}_2] \\ A^* &= \text{diag} [A_1^* \quad A_2^* \quad A_3^*] \\ B^* &= \begin{bmatrix} B_1^* \\ B_2^* \\ B_3^* \end{bmatrix} ; \quad A_k^* = \begin{bmatrix} 0 & 1 \\ -\lambda_k & 0 \end{bmatrix} \\ B_k^* &= \begin{bmatrix} 0 & 0 & \dots & 0 \\ \hat{h}_{k1} & \hat{h}_{k2} & \dots & \hat{h}_{km} \end{bmatrix} \\ k &= 1, 2, 3 \end{aligned} \quad (4.2-37)$$

For symmetric and antisymmetric modes of  $B_k$  appropriate  $\Delta$  and  $\hat{h}_{ki}$  are to be substituted from Table 4.5 and equation (4.2-34). Normalizing the state variables as described in section 3.3.7, we obtain the final state space form corresponding to the orthogonal modes  $\xi$  as

$$\dot{X} = Ax + Bu$$

(4.2-38)

where

$$A = D_N^{-1} A^* D_N = \text{diag} [A_1 \ A_2 \ A_3]$$

$$D_N = \text{diag} [N_0 \ \bar{N}_0 \ N_1 \ \bar{N}_1 \ N_2 \ \bar{N}_2]$$

$$A_1 = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} ; \quad A_2 = \begin{bmatrix} 0 & a_{34} \\ -\frac{\lambda_2}{a_{34}} & 0 \end{bmatrix} ; \quad A_3 = \begin{bmatrix} 0 & a_{56} \\ -\frac{\lambda_3}{a_{56}} & 0 \end{bmatrix}$$

$$a_{12} = \frac{\bar{N}_0}{N_0} ; \quad a_{34} = \frac{\bar{N}_1}{N_1} ; \quad a_{56} = \frac{\bar{N}_2}{N_2}$$

$$B = \{b_{ji}\} = D_N^{-1} B^* = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \frac{1}{N_0} [\hat{h}_{11} & \hat{h}_{12} & \hat{h}_{13} & \dots & \hat{h}_{1m}] \\ 0 & 0 & 0 & \dots & 0 \\ \frac{1}{N_1} [\hat{h}_{21} & \hat{h}_{22} & \hat{h}_{23} & \dots & \hat{h}_{2m}] \\ 0 & 0 & 0 & \dots & 0 \\ \frac{1}{N_2} [\hat{h}_{31} & \hat{h}_{32} & \hat{h}_{33} & \dots & \hat{h}_{3m}] \end{bmatrix}$$

$$b_{2k-1,i} = 0 ; \quad b_{2k,i} = \frac{1}{N_{k-1}} \hat{h}_{ki} , \quad k=1, 2, 3$$

$$i=1, 2, \dots, m$$

$$x_{2k+1} = \frac{x_{2k+1}^*}{N_k} ; \quad x_{2k+2} = \frac{x_{2k+2}^*}{N_k} , \quad k=0, 1, 2 \quad (4.2-39)$$

$N_k, \bar{N}_k$  being the chosen weighting factors. The value of  $\lambda_1$  is zero for both types of modes of  $B_F$  and  $\lambda_2, \lambda_3$  are to be taken from Table 4.5 for the appropriate mode of  $B_F$ . The elements  $\hat{h}_{ki}, k=1, 2, 3, i=1, 2, \dots, m$  are to be similarly substituted from (4.2-34) using (4.2-26),

(4.2-23) and (4.2-24).

Note: This orthogonal form leads to simpler A matrix consisting of  $2 \times 2$  diagonal blocks which simplifies the algebra (the system can be separated into  $N+1$  second order subsystems for computational purposes).



### 4.3 Approximate Degree of controllability

In chapter Two, section 2.4.3, outlined the procedure to obtain the spatial form (Def. 2.3) in real space for the state displacement  $\delta$ . Following this procedure we will first obtain the Jordan canonical form  $J$  of the matrix  $A$  in (4.2-35), and also the transformation matrix  $P$  and  $P^{-1}$ . These are as follows:

$$J = \text{diag} [J_1, i\sqrt{\lambda_2} \quad -i\sqrt{\lambda_2} \quad i\sqrt{\lambda_3} \quad -i\sqrt{\lambda_3}]$$

$$J_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad P = \text{diag} [P_1, P_2, P_3]$$

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a_{12}} \end{bmatrix}; \quad P_2 = \frac{1}{\sqrt{1+c_1^2}} \begin{bmatrix} 1 & 1 \\ ic_1 & -ic_1 \end{bmatrix}; \quad P_3 = \frac{1}{\sqrt{1+c_2^2}} \begin{bmatrix} 1 & 1 \\ ic_2 & -ic_2 \end{bmatrix}$$

$$c_1 = \sqrt{\lambda_2}/a_{34}; \quad c_2 = \sqrt{\lambda_3}/a_{56}$$

$$P^{-1} = \text{diag} [P_1^{-1}, P_2^{-1}, P_3^{-1}]$$

$$P_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & a_{12} \end{bmatrix}; \quad P_2^{-1} = \frac{\sqrt{1+c_1^2}}{2c_1} \begin{bmatrix} c_1 & -i \\ c_1 & i \end{bmatrix}; \quad P_3^{-1} = \frac{\sqrt{1+c_2^2}}{2c_2} \begin{bmatrix} c_2 & -i \\ c_2 & i \end{bmatrix}$$

(4.3-1)

where  $a_{12}$ ,  $a_{34}$ ,  $a_{56}$ , etc. are defined in (4.2-39). The eigenvalues

are 0, 0,  $\pm i\sqrt{\lambda_2}$ ,  $\pm i\sqrt{\lambda_3}$  and the Jordan blocks are  $J_1, J_2, \dots, J_5$  of order  $\nu_x$  equal to 2, 1, 1, 1, 1, respectively. The spatial form is

$$\delta = \int_0^T [\tilde{P}_1 \tilde{P}_2 \dots \tilde{P}_5] \begin{bmatrix} e^{-J_1 t} \hat{P}_1 u(t) \\ e^{-J_2 t} \hat{P}_2 u(t) \\ \vdots \\ e^{-J_5 t} \hat{P}_5 u(t) \end{bmatrix} dt \quad (4.3-2)$$

where

$$\tilde{P}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/a_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{P}_2 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{1+c_1^2} \\ ic_1/\sqrt{1+c_1^2} \\ 0 \\ 0 \end{bmatrix}; \quad \tilde{P}_3 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{1+c_1^2} \\ -ic_1/\sqrt{1+c_1^2} \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{P}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/\sqrt{1+c_2^2} \\ ic_2/\sqrt{1+c_2^2} \end{bmatrix}; \quad \tilde{P}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/\sqrt{1+c_2^2} \\ -ic_2/\sqrt{1+c_2^2} \end{bmatrix}$$

$$\hat{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{P}_2 = \begin{bmatrix} 0 & 0 & \frac{\sqrt{1+c_1^2}}{2} & -\frac{ic_1\sqrt{1+c_1^2}}{2c_1} & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
 \hat{p}_3 &= \begin{bmatrix} 0 & 0 & \frac{\sqrt{1+c_1^2}}{2} & \frac{i\sqrt{1+c_1^2}}{2c_1} & 0 & 0 \end{bmatrix} \\
 \hat{p}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{1+c_2^2}}{2} & -\frac{i\sqrt{1+c_2^2}}{2c_2} \end{bmatrix} \\
 \hat{p}_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{1+c_1^2}}{2} & \frac{i\sqrt{1+c_2^2}}{2c_2} \end{bmatrix}
 \end{aligned} \tag{4.3-3}$$

Now, replace the complex pair of partitions in (4.3-2) by real partitions from (2.4-25) as explained in section 2.4.2, to obtain  $\delta$  in real space, i.e.,

$$\delta = C \int_0^T \alpha(t) dt \tag{4.3-4}$$

where

$$C = [c_1 \ c_2 \ \dots \ c_6] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a_{12}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1+c_1^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c_1}{\sqrt{1+c_1^2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{1+c_2^2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c_2}{\sqrt{1+c_2^2}} \end{bmatrix}$$

$$\alpha(t) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} e^{-\hat{A}_1 t} \hat{P}_1 B u \\ 2(\hat{P}_2^R \cos \sqrt{\lambda_2} t + \hat{P}_2^I \sin \sqrt{\lambda_2} t) B u \\ 2(\hat{P}_2^R \sin \sqrt{\lambda_2} t - \hat{P}_2^I \cos \sqrt{\lambda_2} t) B u \\ 2(\hat{P}_4^R \cos \sqrt{\lambda_3} t + \hat{P}_4^I \sin \sqrt{\lambda_3} t) B u \\ 2(\hat{P}_4^R \sin \sqrt{\lambda_3} t - \hat{P}_4^I \cos \sqrt{\lambda_3} t) B u \end{bmatrix} \quad (4.3-5)$$

in which

$$e^{-\hat{A}_1 t} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}; \quad \begin{aligned} \hat{P}_2 &= \hat{P}_2^R + i \hat{P}_2^I \\ \hat{P}_4 &= \hat{P}_4^R + i \hat{P}_4^I \end{aligned} \quad (4.3-6)$$

Now, since all the Jordan blocks  $J_k$  are associated with distinct eigenvalues,  $\lambda_j = 1$ ,  $j=k=1,2,\dots,5$ , which from theorem 2.3 gives the necessary condition  $m \geq 1$  so that the system may be completely controllable ( $m$  is the dimension of  $u(t)$ ). There is no need to consider the rank test (necessary and sufficient condition) for complete controllability because we are going to proceed with the derivation of the approximate degree of controllability  $\rho^*$ , and from theorem 2.6 we know that the system is uncontrollable if and only if  $\rho^*$  goes to zero.

The r.h.s. of  $\alpha(t)$  in (4.3-5) can be written as

$$\alpha(t) = \text{diag}[\alpha_A \quad \alpha_B \quad \alpha_C] \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u$$

$$= \begin{bmatrix} \alpha_A B_1 u \\ \alpha_B B_2 u \\ \alpha_C B_3 u \end{bmatrix} \quad (4.3-7)$$

where

$$\alpha_A = \begin{bmatrix} 1 & -a_{12}t \\ 0 & a_{12} \end{bmatrix}; \quad \alpha_B = \sqrt{1+c_1^2} \begin{bmatrix} \cos\sqrt{\lambda_2}t & -\frac{1}{c_1}\sin\sqrt{\lambda_2}t \\ \sin\sqrt{\lambda_2}t & \frac{1}{c_1}\cos\sqrt{\lambda_2}t \end{bmatrix}$$

$$\alpha_C = \sqrt{1+c_2^2} \begin{bmatrix} \cos\sqrt{\lambda_3}t & -\frac{1}{c_2}\sin\sqrt{\lambda_3}t \\ \sin\sqrt{\lambda_3}t & \frac{1}{c_2}\cos\sqrt{\lambda_3}t \end{bmatrix}; \quad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$B_k = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \frac{1}{N_{k-1}} [\hat{h}_{k1} & \hat{h}_{k2} & \hat{h}_{k3} & \dots & \hat{h}_{km}] \end{bmatrix}, \quad k=1, 2, 3$$

(4.3-8)

From these, we obtain

$$\alpha_1(t) = -\frac{1}{N_0} t \sum_{i=1}^m \hat{h}_{1i} u_i$$

$$\alpha_2(t) = \frac{1}{N_0} \sum_{i=1}^m \hat{h}_{1i} u_i$$

$$\alpha_3(t) = -\frac{\sqrt{1+c_1^2}}{\sqrt{\lambda_2} N_1} \sin\sqrt{\lambda_2}t \sum_{i=1}^m \hat{h}_{2i} u_i$$

$$\alpha_4(t) = \frac{\sqrt{1+c_1^2}}{\sqrt{\lambda_2} N_1} \cos\sqrt{\lambda_2}t \sum_{i=1}^m \hat{h}_{2i} u_i$$

$$\alpha_5(t) = -\frac{\sqrt{1+c_2^2}}{\sqrt{\lambda_3} N_2} \sin\sqrt{\lambda_3}t \sum_{i=1}^m \hat{h}_{3i} u_i$$

$$\alpha_6(t) = \frac{\sqrt{1+c_2^2}}{\sqrt{\lambda_3} N_2} \cos \sqrt{\lambda_3} t \sum_{i=1}^m \hat{h}_{3i} u_i \quad (4.3-9)$$

where  $a_{12}$ ,  $a_{34}$ ,  $a_{56}$ ,  $c_1$ ,  $c_2$  have been substituted from (4.2-33) and (4.3-1).

The approximate recovery region  $R^*$  (6 dimensional parallel-omiped) is given, according to (2.6-3), as

$$R^* = [\gamma_1 c_1, \gamma_2 c_2, \dots, \gamma_6 c_6] \quad (4.3-10)$$

where, as in (2.6-2),

$$\gamma_j = \max_{i=1, 2, \dots, m} \int_0^T \alpha_j(t) dt, \quad t \in [0, T], \quad |u_i| = 1 \quad (4.3-11)$$

and from (2.6-9),

$$(C^T)^{-1} = [\gamma_1 \quad \gamma_2 \quad \dots \quad \gamma_6] \\ \|\gamma_j\| = \sqrt{\gamma_j^T \gamma_j}, \quad j = 1, 2, \dots, 6 \quad (4.3-12)$$

From (4.3-5), evaluating  $(C^T)^{-1}$  we obtain,

$$\|\gamma_1\| = 1; \quad \|\gamma_2\| = a_{12}; \quad \|\gamma_3\| = \sqrt{1+c_1^2} \\ \|\gamma_4\| = \frac{\sqrt{1+c_1^2}}{c_1}; \quad \|\gamma_5\| = \sqrt{1+c_2^2}; \quad \|\gamma_6\| = \frac{\sqrt{1+c_2^2}}{c_2} \quad (4.3-13)$$

Applying the maximization to  $\alpha_j(t)$  in (4.3-9), we can obtain:

$$\begin{aligned}
\gamma_1 &= \frac{1}{2} \frac{T^2}{N_0} \sum_{i=1}^m |h_{1i}| \\
\gamma_2 &= \frac{T}{N_0} \sum_{i=1}^m |\hat{h}_{1i}| \\
\gamma_3 &= \frac{\sqrt{1+c_1^2}}{\sqrt{\lambda_2} N_1} \left\{ \int_0^T |\sin \sqrt{\lambda_2} t| dt \right\} \sum_{i=1}^m |\hat{h}_{2i}| \\
\gamma_4 &= \frac{\sqrt{1+c_1^2}}{\sqrt{\lambda_2} N_1} \left\{ \int_0^T |\cos \sqrt{\lambda_2} t| dt \right\} \sum_{i=1}^m |\hat{h}_{2i}| \\
\gamma_5 &= \frac{\sqrt{1+c_2^2}}{\sqrt{\lambda_3} N_2} \left\{ \int_0^T |\sin \sqrt{\lambda_3} t| dt \right\} \sum_{i=1}^m |\hat{h}_{3i}| \\
\gamma_6 &= \frac{\sqrt{1+c_2^2}}{\sqrt{\lambda_3} N_2} \left\{ \int_0^T |\cos \sqrt{\lambda_3} t| dt \right\} \sum_{i=1}^m |\hat{h}_{3i}|
\end{aligned} \tag{4.3-14}$$

The normal distances to the surfaces of the parallelopiped region are obtained using (2.6-11), i.e.,

$$d_j = \frac{\gamma_j}{\|g_j\|}, \quad j = 1, 2, \dots, 6 \tag{4.3-15}$$

where  $\gamma_j$ ,  $\|g_j\|$  are given in (4.3-14), (4.3-13), respectively, and  $\hat{h}_{ki}$  are given by (4.2-34).

Assuming that  $T$  is long compared with the period of the sine wave (cosine wave) (so that the effect of partial completion of the final period of oscillation of the sine wave is negligible), the absolute value of the sine or cosine wave can be replaced by  $2/\pi$ , its average over a period. With this substitution in (4.3-14) and substituting for  $\|g_j\|$  from (4.3-13), (4.3-15) becomes

$$d_1 = \frac{T^2}{2N_0} \sum_{i=1}^m |\hat{h}_{1i}|$$

$$d_2 = \frac{T}{N_0} \sum_{i=1}^m |\hat{h}_{1i}|$$

$$d_3 = \frac{2T}{\pi N_1} \frac{1}{\sqrt{\lambda_2}} \sum_{i=1}^m |\hat{h}_{2i}|$$

$$d_4 = \frac{2T}{\pi N_1} \sum_{i=1}^m |\hat{h}_{2i}|$$

$$d_5 = \frac{2T}{\pi N_2} \frac{1}{\sqrt{\lambda_3}} \sum_{i=1}^m |\hat{h}_{3i}|$$

$$d_6 = \frac{2T}{\pi N_2} \sum_{i=1}^m |\hat{h}_{3i}|$$

(4.3-16)

in which (4.3-1) and (4.2-38) are used for  $c_1$ ,  $c_2$ ,  $a_{12}$ ,  $a_{34}$ , and  $a_{56}$ .

The approximate degree of controllability  $\rho^*$  based on the parallelepiped bound can now be written, according to (2.6-12), as

$$\rho^* = \min_j d_j, \quad j = 1, 2, \dots, 6 \quad (4.3-17)$$

where the  $d_j$  are given by (4.3-16). To evaluate  $\rho^*$  we need to compute  $|\hat{h}_{\kappa i}|$ ,  $\kappa=1,2,3$ ,  $i=1,2,\dots,m$ , which are given in (4.2-34). The information on the type and strength of actuators and their locations is contained in  $\hat{h}_{\kappa i}$ . To put (4.3-16) in a more elegant form, define, in (4.2-34), the following:

$$\begin{aligned} \Gamma_{F\kappa}(\gamma_i^*) &= \hat{v}_{1\kappa} \gamma_{1i}^* + \hat{v}_{2\kappa} w_1^{(i)} + \hat{v}_{3\kappa} w_2^{(i)} \\ \Gamma_{c\kappa}(\gamma_i^*) &= \hat{v}_{1\kappa} + \hat{v}_{2\kappa} z_1^{(i)} + \hat{v}_{3\kappa} z_2^{(i)} \end{aligned} \quad (4.3-18)$$

$$\kappa = 1, 2, 3; \quad i = 1, 2, \dots, m$$

where  $i$  refers to a particular actuator. If we pick a certain location



$r_i^*$  for some actuator  $i=1$  then the right side of (4.3-18) is a number. Although we will use a finite number of actuators at discrete locations we can pick any value for  $r_i^*$  within the available range since our model is a continuum. Hence, the index  $i$  can be dropped in the above and  $r$  considered a continuous variable. Also, as mentioned in section 4.2,  $r_{11}^*$  and  $r_i^*$  can be considered as one variable since the change of location from the rigid body to the flexible body is taken care of by  $\delta_F$  in  $x_i$  and  $z_i$ . Thus, there is one continuous variable  $r^*$  in  $\hat{b}_1$  direction. The functions in (4.3-18) then can be written as

$$\begin{aligned}\Gamma_{Fk}(\gamma^*) &= \tilde{V}_{1k} \gamma^* + \tilde{V}_{2k} w_1 + \tilde{V}_{3k} w_2 \\ \Gamma_{ck}(\gamma^*) &= \tilde{V}_{1k} + \tilde{V}_{2k} z_1 + \tilde{V}_{3k} z_2\end{aligned}\quad (4.3-19)$$

$k = 1, 2, 3$

where

$$\begin{aligned}W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= \begin{bmatrix} 0.98565902 \phi_1^*(\gamma^*) \delta_{F\gamma} - 0.00071014 \\ 0.98530145 \phi_2^*(\gamma^*) \delta_{F\gamma} - 0.00039314 \end{bmatrix} \\ Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} 0.98565902 \phi_1^{*'}(\gamma^*) \delta_{F\gamma} \\ 0.98530145 \phi_2^{*'}(\gamma^*) \delta_{F\gamma} \end{bmatrix}\end{aligned}$$

for symmetric mode of  $\Theta_F$

$$W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0.98538098 \phi_1^*(\gamma^*) \delta_{F\gamma} \\ 0.98521622 \phi_2^*(\gamma^*) \delta_{F\gamma} \end{bmatrix}$$

$$\bar{z} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = \begin{bmatrix} 0.98538098 & \phi_1^{*'}(\gamma^*) \delta_{F\gamma} \\ 0.98521622 & \phi_2^{*'}(\gamma^*) \delta_{F\gamma} \end{bmatrix}$$

for antisymmetric mode of  $\Theta_F$

$$\phi_j^{*'}(\gamma^*) = \begin{cases} \hat{\phi}_j^{*'}(x^*) & , -1 \leq x^* \leq 0 \\ \hat{\phi}_j^{*'}(x^*) & , 0 \leq x^* \leq 1 \end{cases}$$

$$\tilde{\phi}_j^{*'}(x^*) = \delta \hat{\phi}_j^{*'}(-x^*)$$

$$\phi_j^{*'}(\gamma^*) = \begin{cases} \tilde{\phi}_j^{*'}(x^*) & , -1 \leq x^* \leq 0 \\ \hat{\phi}_j^{*'}(x^*) & , 0 \leq x^* \leq 1 \end{cases}$$

$$\tilde{\phi}_j^{*'}(x^*) = -\delta \hat{\phi}_j^{*'}(s^*) , \quad 0 \leq s^* \leq 1 , \quad s^* = -x^*$$

$j = 1, 2$

$$\delta = \begin{cases} 1 & \text{for symmetric mode of } \Theta_F \\ -1 & \text{for antisymmetric mode of } \Theta_F \end{cases}$$

$$\delta_{F\gamma} = \begin{cases} 0 & \gamma^* \in \Theta_R \\ 1 & \gamma^* \in \Theta_F \end{cases} \quad (4.3-20)$$

In the above the index  $i$  which refers to a particular actuator and its location along  $r^*$  has been dropped. The functions  $\Gamma_F$  correspond to force type actuators and the functions  $\Gamma_c$  correspond to torque type actuators. Let

$$\begin{aligned} \Gamma_F^{(i)} &= \Gamma_F(\gamma_i^*) ; \quad \Gamma_{Fk}^i = \Gamma_{Fk}(\gamma_i^*) \\ \Gamma_c^i &= \Gamma_c(\gamma_i^*) ; \quad \Gamma_{ck}^i = \Gamma_{ck}(\gamma_i^*) \end{aligned} \quad (4.3-21)$$

$k = 1, 2, 3$

Then from (4.2-34) and (4.3-18)

$$|\hat{h}_{ki}| = \begin{cases} |k_{Fi} \Gamma_{Fk}^i| & , i \in \mathcal{F} \text{ (force type)} \\ |k_{Ci} \Gamma_{Ck}^i| & , i \in \mathcal{C} \text{ (torque type)} \end{cases} \quad (4.3-22)$$

$k = 1, 2, 3$

We will take the constants  $k_{Fi}$ ,  $k_{Ci}$  to be positive (see (4.2-19))

which then leads to

$$|\hat{h}_{ki}| = \begin{cases} k_{Fi} |\Gamma_{Fk}^i| & \text{for } i \in \mathcal{F} \\ k_{Ci} |\Gamma_{Ck}^i| & \text{for } i \in \mathcal{C} \end{cases} \quad (4.3-23)$$

$k = 1, 2, 3$

and

$$\sum_{i=1}^m |\hat{h}_{ki}| = \sum_{j \in \mathcal{F}} k_{Fj} |\Gamma_{Fk}^j| + \sum_{j \in \mathcal{C}} k_{Cj} |\Gamma_{Ck}^j| \quad (4.3-24)$$

$k = 1, 2, 3$

These are to be substituted in (4.3-15) to compute the  $d_j$  whose minimum value, according to (4.3-17), is the approximate degree of controllability  $\rho^*$  of the system.

In the next chapter we will discuss the results and examine the effect of actuator locations on the degree of controllability of the system.

## 5. RESULTS AND DISCUSSIONS

### 5.1 Introduction

In chapter Four we derived expressions for the approximate degree of controllability  $\rho^*$  based on a specific model of a large flexible spacecraft. In this chapter we will discuss the results for single and multiple actuator distributions and examine their effect on the approximate degree of controllability  $\rho^*$ .

## 5.2 Influence Curves for Actuator Locations

The expressions for the normal distances  $d_j$  in (4.3-15) can be written essentially as products of three types of functions  $f_1$ ,  $f_2$ , and  $f_3$  where  $f_1$  is a function of total time  $T$ ,  $f_2$  is a function of weighting factor and  $f_3$  is a function of actuator strength and the effect due to its location in the system. Or, a typical normal distance  $d$  can be written as

$$d = f_1 f_2 f_3 \quad (5.2-1)$$

where

$$\begin{aligned} f_1 &= f_1(T) \\ f_2 &= \begin{cases} f_2(N, \sqrt{\lambda}) \\ \text{or} \\ f_2(\bar{N}) \end{cases} \\ f_3 &= f_3(k_F, k_C, |\Gamma_F|, |\Gamma_C|) \end{aligned} \quad (5.2-2)$$

It is not important how we define  $f_1$ ,  $f_2$ ,  $f_3$ ; the only significance is that  $T$  belongs to  $f_1$ , the weighting factor belongs to  $f_2$ , and the terms related to the actuators belong to  $f_3$ . Hence, in (4.3-15), noting (4.3-24) choose

$$\begin{aligned} f_3^k &= f_3^k(k_F, k_C, |\Gamma_F|, |\Gamma_C|) \\ &= \sum_{i=1}^m |\hat{h}_{ki}| = \sum_{j \in \mathcal{F}} k_{Fj} |\Gamma_{Fk}^j| + \sum_{j \in \mathcal{C}} k_{Cj} |\Gamma_{Ck}^j| \end{aligned} \quad (5.2-3)$$

$k = 1, 2, 3$

Then, the normal distances  $d_j$  in (4.3-15) become

$$d_{2k-1} = f_1^{2k-1} f_2^{2k-1} f_3^k$$

$$d_{2k} = f_1^{2k} f_2^{2k} f_3^k$$

$$k = 1, 2, 3$$

$$(5.2-4)$$

where  $f_3^k$  is defined in (5.2-3) and the rest need not be defined precisely at this stage. But we will include the  $\lambda$  term in  $f_2$  so that the dimensions of  $f_2$  will be the same in either form (see (5.2-2)). The factor  $\lambda$  is known because it is a system constant. Hence, the unknowns in  $d_j$  are total time  $T$ , weighting factors  $N$ ,  $\bar{N}$ , normalized actuator strengths  $k_f$ ,  $k_e$ , and the influence factors  $|f_f|$ ,  $|f_e|$  due to the locations of actuators. All of these must be known before  $\rho^*$  can be computed.

If we are given a distribution of actuators we can presume the type of actuator (force type or torque type) at any given location and also its strength ( $k_f$  or  $k_e$ ). From the knowledge of only the type of actuator at any given location  $r^*$  the  $|f|$  ( $|f_f|$  or  $|f_e|$ ) factors are determined. With the additional knowledge of the strength of these actuators the function  $f_3$  is determined. It is reasonable to presume that we may not want to change the weighting factors often. Hence,  $f_2$  could also be determined if we decide on a set of weighting factors for the system states. This leaves total time  $T$  as the only remaining variable of  $d_j$ , so  $d_j$  can be plotted as functions of total time  $T$ . And hence,  $\rho^*$  can be evaluated for any  $T$ .

The procedure to determine the approximate degree of controllability  $\rho^*$  can be summarized in the following steps:

- 1) Choose the locations  $r_1^*, r_2^*, \dots, r_m^*$  where the actuators are to be located. Classify them into the two groups, one for the force type actuators and the other for the torque type actuators.
- 2) Tabulate  $|\Gamma_{fk}|$  and  $|\Gamma_{ck}|$  values for these locations corresponding to the appropriate group and for the type of mode of motion of  $B_F$  (symmetric or antisymmetric).
- 3) Compute the contribution of each actuator by taking the product of its strength ( $k_F, k_C$ ) and the appropriate  $|\Gamma|$  value tabulated in step 2.
- 4) Take the sum of all the contributions in step 3 to obtain the function

$$f_j^k = \sum_{i=1}^m |\hat{h}_{ki}|$$

- 5) Choose a set of weighting factors  $N_i, \bar{N}_i$  and compute the functions

$$f_2^{2i+1} = f_2^{2i+1}(N_i, \sqrt{\lambda})$$

$$f_2^{2i+2} = f_2^{2i+2}(\bar{N}_i)$$

$$i = 0, 1, 2$$

in the expressions for  $J_j, j=1, 2, \dots, 6$ .

- 6) Given a total time  $T$  compute  $f_1^j = f_1^j(T)$  in the expressions for  $d_j, j=1, 2, \dots, 6$ .

7) From these evaluate the normal distances  $d_j$  as follows:

$$d_{2k-1} = f_1^{2k-1} f_2^{2k-1} f_3^k$$

$$d_{2k} = f_1^{2k} f_2^{2k} f_3^k, \quad k = 1, 2, 3$$

8) The approximate degree of controllability  $\rho^*$  of the system based on the given distribution of actuators is the minimum of all the normal distances  $d_j$  evaluated in step 7.

Thus, by the above procedure  $\rho^*$  can be evaluated for various distributions of actuators (type, strength and locations known), and these distribution patterns can be ranked in descending order of  $\rho^*$ , the higher the  $\rho^*$  the better the corresponding distribution pattern. The decision as to which of a given set of distributions is better may or may not depend on  $T$ . If a limited range for  $T$  is prescribed for control purposes (this may be the case in spacecraft control) then it is possible to decide from the  $\rho^*$  curves the distribution that is best on the average over this range of  $T$ .

At this point we can say our objective of ranking various distributions is fulfilled. However, we could gain a little more insight into the behavior of  $\rho^*$  from the point of view of location of actuators from what we will call influence curves ( $|\Gamma|$  curves) for actuator locations.

By  $|\Gamma|$  curves we mean the two sets of curves  $|\Gamma_F|$  and  $|\Gamma_C|$ . The  $|\Gamma_F|$  curves belong to the force type of actuators and the  $|\Gamma_C|$  curves belong to the torque type of actuators. (Note: these are separate sets of curves for the symmetric and antisymmetric modes of motion of  $B_F$ , these two types of modes being treated separately.) The  $|\Gamma|$  curves are



functions of only the actuator location  $r^*$ . A  $|f|$  ( $|f_F|$  or  $|f_C|$ ) set consists of  $n/2$  curves  $|f_k|$  ( $|f_{Fk}|$  or  $|f_{Ck}|$ ),  $k=1,2,\dots,n/2$ , where  $n$  is the order of the system, i.e., there are as many component curves of  $|f|$  as there are position variables (as opposed to velocity variables) in the original state vector (not normalized). (For our model  $n=6$ .) Each  $|f_k|$  curve corresponds to a particular mode (in this case vehicle normal mode). But the distances  $l_j$  need not in general correspond to any mode in the original system, because the  $l_j$  are along normals (to the surfaces of a parallelepiped region  $R^*$ ) whose orientation in the state space depends on the shape of  $R^*$ .

Figures 8 through 11 show these  $|f|$  curves. Figures 8 and 10 represent the sets of  $|f_F|$  curves for force type actuators for the symmetric mode and antisymmetric mode of motion of  $B_F$ , respectively. Figures 9 and 11 represent the sets of  $|f_C|$  curves for torque type actuators for the symmetric mode and antisymmetric mode of motion of  $B_F$ . These curves are plotted as functions of the absolute value  $|r^*|$  of the actuator location. The absolute value is used because these curves are duplicated for  $\pm r^*$  (the appendages are identical and we have a symmetric physical system). The location  $r^*$  is measured from the point C (see Fig. 5) along  $\hat{b}_1$  direction. For locations  $0 \leq |r^*| \leq r_C^*$ ,  $r_C^* = 0.03333333$ , the actuator is on the rigid body  $B_R$  and for locations  $r_C^* < |r^*| \leq (1 + r_C^*)$  the actuator is on the flexible body  $B_F$  on either of the two appendages. The absolute value  $|r^*|$  is plotted along the x-axis and the  $|f_{Fk}|$  or  $|f_{Ck}|$  are plotted along the y-axis.

From (4.3-23) it can be seen that the contribution due to each

actuator is added up measured in terms of absolute values to obtain the  $d_i$ . Moreover, the contribution of every actuator is a product of two terms, one involving only the strength of the actuator ( $k_F$ ,  $k_L$ ) and the other involving only the effect due to its location in the system. The influence curves ( $|f|$  curves) are very useful in computing the contribution to the degree of controllability  $\rho^*$  of each actuator. Because these curves, being independent of actuator strengths, can be used for different actuators at the same locations, and the total contribution of a new set of actuators is obtained simply by adding up the products of the new actuator strengths and the same influence factors for the old set.

In the next several sections we will examine various options of the control actuators and their bearing on the approximate degree of controllability  $\rho^*$  of the system. These options will be examined for both types of modes of motion of  $B_F$ , symmetric and antisymmetric. These options are as follows:

- 1) Single force actuator
- 2) Single torque actuator
- 3) Multiple Actuators

## 5.3 Single Force Actuator

### 5.3.1 Introduction

The normal distances  $d_j$ , from (4.3-15) ( $m = 1$  in this case), are given by

$$\begin{aligned} d_1 &= \left( \frac{T^2}{2N_0} k_{F1} \right) | \Gamma_{F1}' | \\ d_2 &= \left( \frac{T}{N_0} k_{F1} \right) | \Gamma_{F1}' | \\ d_3 &= \left( \frac{2T}{\pi N_1 \sqrt{\lambda_2}} k_{F1} \right) | \Gamma_{F2}' | \\ d_4 &= \left( \frac{2T}{\pi N_1} k_{F1} \right) | \Gamma_{F2}' | \\ d_5 &= \left( \frac{2T}{\pi N_2 \sqrt{\lambda_3}} k_{F1} \right) | \Gamma_{F3}' | \\ d_6 &= \left( \frac{2T}{\pi N_2} k_{F1} \right) | \Gamma_{F3}' | \end{aligned} \quad (5.3-1)$$

where (4.3-24) has been used. For any chosen set of weighting factors  $N_0$ ,  $N_1$ ,  $N_2$ , for any total time  $T$  and for any strength of the force actuator each  $d_j$  is proportional to the height of one of the three curves of  $| \Gamma_F' |$  (Fig. 5 for symmetric mode and Fig. 10 for antisymmetric mode). Thus, if  $d_j$  were to be plotted as a function of location  $|r^*|$  we would have six curves, a pair of them proportional to each  $| \Gamma_{Fk}' |$ . Because of this proportionality the maximum, minimum and zero values of any  $d_j$  curve correspond to the maximum, minimum and zero values of its corresponding  $| \Gamma_{Fk}' |$  curve, i.e., the behavior of a  $| \Gamma_{Fk}' |$  curve is the same as the

behavior of its corresponding pair of  $d_j$  curves with respect to these values. Therefore, we can examine the  $|\Gamma_{F_k}^1|$  curves to gain some insight into the behavior of the approximate degree of controllability  $\rho^*$  as a function of location  $|r^*|$  of the actuator.

### 5.3.2 Symmetric Mode of $B_F$

Figure 3 shows the set of  $|\Gamma_F^1|$  curves for the symmetric mode of motion of  $B_F$ . There are four locations of  $|r^*|$  at which  $\rho^*$  goes to zero indicating the system is uncontrollable at these locations. The  $|\Gamma_{F_1}^1|$  and  $|\Gamma_{F_2}^1|$  curves have one such location each and the  $|\Gamma_{F_3}^1|$  curve has two such locations. These locations are tabulated in Table 5. Recall that for the symmetric case the vehicle normal modes are the same as the hybrid modes  $\tilde{\eta}(\theta, \eta)$ , and hence the zero of any  $|\Gamma_{F_k}^1|$  indicates that the corresponding hybrid mode is uncontrollable. At  $|r^*| = 0$  (point C) the rigid body mode  $\theta$  is uncontrollable but the appendage modes  $\eta_1, \eta_2$  are controllable. At this location the actuator can generate no torque on the vehicle in the inertial system, so  $\theta$  is uncontrollable. The symmetric motion of  $B_F$  (Fig. 5) produces only translatory motion of  $B_R$  and by controlling the  $\eta$  modes this can be killed. But if the disturbance includes  $\theta$  this cannot be controlled. The locations on  $B_F$  at which  $|\Gamma_{F_1}^1|, |\Gamma_{F_2}^1|$  are zero are inertial nodes (obtained by putting  $\dot{\underline{r}}_R = 0$ ) for the corresponding modes (first and second modes) of  $B_F$ . At an inertial node the effect of the force of the actuator is not felt by the corresponding mode, and hence its shape cannot be controlled.

Having marked the zeroes of the  $|\Gamma_F^i|$  curves let us examine how they behave over the range of  $|r^*|$ . For any location  $|r^*|$  on the rigid body  $B_R$  ( $0 \leq |r^*| \leq r_c^A$ )  $|\Gamma_{F_1}^i|$  and  $|\Gamma_{F_3}^i|$  are constant (Fig. 9), hence no improvement is attained in the controllability of the two modes  $\eta_1$  and  $\eta_2$ . On the flexible body  $B_F$  for locations  $|r^*|$  beyond the zero of  $|\Gamma_{F_1}^i|$  ( $|r^*| \approx 0.05$ ) up to the crest of  $|\Gamma_{F_3}^i|$  ( $|r^*| \approx 0.5$ ) all three  $|\Gamma_{F_k}^i|$  curves (see Fig. 8) increase monotonically. Likewise for locations beyond the second zero of  $|\Gamma_{F_3}^i|$  ( $|r^*| \approx 0.92$ ) up to the tip of the appendages all the three  $|\Gamma_{F_k}^i|$  curves increase monotonically. Hence, over these two ranges,  $0.05 \leq |r^*| \leq 0.5$  and  $|r^*| \geq 0.92$  the degree of controllability  $\rho^*$  of the system improves as  $|r^*|$  increases. Moreover, the individual maximum of each  $|\Gamma_{F_k}^i|$  curve occurs at the tip of the appendages, and hence  $\rho^*$  is maximum for the location at the tip of the appendages.

In conclusion we can state the following:

For the symmetric mode of motion of  $B_F$ , if a single force actuator is to be used for attitude and shape control the best location for the actuator, from the controllability point of view, is the tip of the appendages regardless of the strength of the actuator, the total time  $T$  allowed for control, and any set of weighting factors one might wish to choose for the normal modal coordinates.

Other salient features for the force actuator for symmetric motion of  $B_F$  are:

- 1) Locations in the neighborhood of the zeroes of the influence curves ( $|\Gamma_F^i|$  curves): Since at the zeroes of  $|\Gamma_F^i|$  curves the degree of controllability  $\rho$  of the system goes to zero,

locations in the neighborhood of these zeroes exhibit a very poor quality of controllability of the system and should be avoided as far as possible.

- 2) Two broad distinct ranges of  $|r^*|$  on  $B_F$ ,  $0.05 \leq |r^*| \leq 0.5$  and  $|r^*| \geq 0.82$ : Over these ranges controllability of the system improves steadily ( $\rho^*$  increases) as  $|r^*|$  increases.
- 3) Controllability of the  $\theta$  mode (rigid body mode) improves steadily as  $|r^*|$  increases over its entire range in the system ( $0 \leq |r^*| \leq (1 + \xi_0^*)$ ).
- 4) The quality of controllability for the two modes  $\eta_1$  and  $\eta_2$  remains unchanged for locations on the rigid body  $B_R$  ( $0 \leq |r^*| \leq r_0^*$ ).

These features can be usefully applied when searching for other suitable locations for the force actuator than the tip of the appendages. Such occasions might arise due to physical restrictions and other practical considerations. In practice one would have to consider a sufficient number of modes in the model to give a good representation of the system dynamic behavior before the actuator location decision can be made.

Thus, for the case of symmetric mode of motion of  $B_F$  we have some valuable information regarding the quality of locations  $|r^*|$  in the system when a force actuator is to be used.

### 5.3.3 Antisymmetric mode of $B_F$

Figure 10 shows the set of  $|\rho_F^*|$  curves for the antisymmetric mode of motion of  $B_F$ . The zeroes of the  $|\rho_F^*|$  curves are tabulated in Table 5. For the antisymmetric modes of  $B_F$  the vehicle normal modes are different from the hybrid modes  $\tilde{\eta}$  ( $\theta, \eta$ ). But  $\xi_0$  is similar to the

rigid body mode  $\theta$  and  $\xi_1$  and  $\xi_2$  are similar to the appendage modes  $\eta_1, \eta_2$ . There are four distinct locations  $|r^*|$  at which at least one  $|\Gamma_{pk}|$  goes to zero indicating system uncontrollability. At  $|r^*| = 0$  (point C) all  $|\Gamma_p|$  curves are zero which indicates none of the three modes is controllable. As stated in section 5.3.2, at this location the force actuator cannot generate any torque on the vehicle in the inertial system, and hence  $\theta$  is uncontrollable. The antisymmetric motion of  $B_F$  (Fig. 7) produces only rotary motion of  $B_R$ , and since this is uncontrollable at  $|r^*| = 0$  the appendage modes  $\eta_1, \eta_2$  are also uncontrollable. The locations on  $B_F$  at which  $|\Gamma_{p_2}|, |\Gamma_{p_3}|$  are zero are inertial nodal points (obtained by putting  $\dot{\mathbf{r}} = 0$ ) for the corresponding vehicle normal modes (the location  $|r^*| = 0$  is also inertially at rest). At the inertial nodes the effect of the actuator force is not felt by the corresponding normal modes (hence, it is uncontrollable).

As in section 5.3.2 we will now examine the nature of the  $|\Gamma_p|$  curves. For locations in the range  $0 \leq |r^*| \leq r_c^*$  (in the rigid body  $B_R$ ) the  $|\Gamma_p|$  curves are all linear which indicates improvement of controllability for all the modes away from  $|r^*| = 0$  up to the root of the appendages. On the flexible body  $B_F$  for locations beyond the zero of  $|\Gamma_{p_2}|$  ( $|r^*| \approx 0.26$ ) up to the crest of  $|\Gamma_{p_3}|$  ( $|r^*| \approx 0.5$ ), and beyond the zero of  $|\Gamma_{p_3}|$  ( $|r^*| \approx 0.32$ ) up to the tip of the appendages all the three  $|\Gamma_{pk}|$  curves increase monotonically. Hence, over these ranges,  $0.26 \leq |r^*| \leq 0.5$  and  $|r^*| \geq 0.82$ , the approximate degree of controllability  $\rho^*$  of the system improves as  $|r^*|$  increases. Again, as in section 5.3.2, the individual maximum of each  $|\Gamma_{pk}|$  curve occurs at

the tip of the appendages. Therefore,  $\rho^*$  is maximum for the location at the tip of the appendages.

The conclusion here, for the antisymmetric case, is the same as that for the case of symmetric mode of motion of  $B_F$ , in section 5.3.2. The tip of the appendages is the best location for the actuator for attitude and shape control regardless of the actuator strength, total time  $T$  and weighting factors for the normal modes.

Other salient features for the force actuator for antisymmetric motion of  $B_F$  are:

- 1) Locations in the neighborhood of the zeroes of the influence curves ( $|I_F^*|$  curves): Same comments as for the case of symmetric motion of  $B_F$ , in section 5.3.2, apply.
- 2) Two broad distinct ranges of  $|r^*|$  on  $B_F$ ,  $0.25 \leq |r^*| \leq 0.5$  and  $|r^*| \geq 0.82$ : Same comments as for the case of symmetric motion of  $B_F$  apply.
- 3) Controllability of all the normal modes improves steadily as  $|r^*|$  increases over the range of locations on the rigid body  $B_R$  ( $0 \leq |r^*| < r_e^*$ ).

As mentioned before these features can be usefully applied in the search for other suitable locations in the system for the force actuator than the tip of the appendages.

Thus, here also for the case of antisymmetric mode of motion of  $B_F$ , some valuable information regarding the quality of locations  $|r^*|$  in the system is obtained with only the knowledge that a force actuator is to be used.



Summarizing, for a single force actuator control we can say that the tip of the appendages is identified as the best location for attitude and shape control for symmetric and antisymmetric modes of motion of  $B_F$ . This conclusion is independent of the strength of the actuator to be used, the total time  $T$  allowed for control, and the weighting factors prescribed for the modes. Additional information is obtained which can be useful in searching for other suitable locations in the system. The information on regions where an actuator should not be located, and the information on distinct ranges for locations  $|r^*|$  where improvement of controllability is guaranteed as  $|r^*|$  increases are very useful. For the symmetric case, some understanding of the quality of controllability is gained for the rigid body mode  $\theta$  and the appendage modes  $\eta_1, \eta_2$ . The information for both the symmetric and antisymmetric modes of motion of  $B_F$  taken together provides a better understanding of the system controllability, and identifies certain narrow ranges for locations  $|r^*|$  that will be suitable as alternatives for the force actuator in order to effectively control both attitude and shape in both types of motion of  $B_F$ .

## 5.4 Single Torque Actuator

### 5.4.1 Introduction

The normal distances  $d_j$ , from (4.3-15) ( $m = 1$  in this case), are given by

$$\begin{aligned}
 d_1 &= \left( \frac{T^2}{2N_0} k_{c1} \right) |\Gamma_{c1}'| \\
 d_2 &= \left( \frac{T}{N_0} k_{c1} \right) |\Gamma_{c1}'| \\
 d_3 &= \left( \frac{2T}{\pi N_1} \frac{1}{\sqrt{\lambda_2}} k_{c1} \right) |\Gamma_{c2}'| \\
 d_4 &= \left( \frac{2T}{\pi N_1} k_{c1} \right) |\Gamma_{c2}'| \\
 d_5 &= \left( \frac{2T}{\pi N_2 \sqrt{\lambda_3}} k_{c1} \right) |\Gamma_{c3}'| \\
 d_6 &= \left( \frac{2T}{\pi N_2} k_{c1} \right) |\Gamma_{c3}'|
 \end{aligned} \tag{5.4-1}$$

where (4.3-24) has been used. This is similar to the case of a single force actuator in section 5.3.1. The  $d_j$  are proportional to the  $|\Gamma_c'|$  curves. The rest of the argument is exactly the same as for a single force actuator except replace  $k_{F1}$  by  $k_{c1}$  (the strength of the actuator) and  $|\Gamma_{Fk}'|$  by  $|\Gamma_{ck}'|$  in section 5.3.1, and refer to Fig. 9 for symmetric motion of  $B_F$  and Fig. 11 for antisymmetric motion of  $B_F$ .

### 5.4.2 Symmetric Mode of $B_F$

Figure 9 shows the set of  $|\Gamma_c'|$  curves for the symmetric mode of

motion of  $B_F$ . The zeroes of these curves are tabulated in Table 5. For the symmetric motion of  $B_F$  the vehicle normal modes are the same as the hybrid modes  $\tilde{\eta}(\theta, \eta)$ , and hence the zeroes of the  $|\Gamma_c^1|$  curves indicate that the corresponding hybrid modes are uncontrollable at those locations. One feature that is different from the force actuator case is that the two curves,  $|\Gamma_{c2}^1|$  and  $|\Gamma_{c3}^1|$  are zero over the entire rigid body range of locations, which indicates the appendage modes  $\eta_1, \eta_2$  are uncontrollable with a torque actuator located in the rigid body  $B_R$ . The  $\theta$  mode is controllable. As before, it is to be remembered that the symmetric motion of  $B_F$  (Fig. 5) produces only translatory motion of  $B_R$ , hence a torque on  $B_R$  is useless in killing the motion of  $B_F$ . At the locations on  $B_F$  at which  $|\Gamma_{c2}^1|, |\Gamma_{c3}^1|$  are zero the inertial slopes are constant for the corresponding modes (these locations are obtained by putting the inertial angular velocity  $\Omega$  of an element on  $B_F$  equal to zero). In this case, the roots of the appendages happen to be such locations of constant inertial slopes for both  $|\Gamma_{c2}^1|$  and  $|\Gamma_{c3}^1|$  curves. And these locations are included in the range of locations over  $B_R$ . There is only one other location on  $B_F$  for a zero of the  $|\Gamma_c^1|$  curves, and that is for the  $|\Gamma_{c3}^1|$  curve. At the locations of constant inertial slopes the effect of the torque of the actuator is not felt by the corresponding modes, and hence their shape cannot be controlled.

The behavior of these curves is more uniform than those for the force actuator. The  $|\Gamma_{c1}^1|$  curve is constant throughout the range of  $|r^*|$ , so no improvement is attained in the controllability of the  $\theta$  mode. The appendage modes are uncontrollable over the entire rigid body range. For locations  $|r^*|$  from the root ( $r_c^*$ ) of an appendage up to the

crest of  $|\Gamma_{c2}|$  ( $|r^*| \approx 0.21$ ), and beyond the zero of  $|\Gamma_{c2}|$  ( $|r^*| \approx 0.51$ ) up to the tip of the appendages, the two curves  $|\Gamma_{c2}|$ ,  $|\Gamma_{c3}|$  increase monotonically while the curve  $|\Gamma_{c1}|$  is unaffected. Hence, over the ranges  $r_c^* \leq |r^*| \leq 0.21$  and  $|r^*| \geq 0.51$ , the approximate degree of controllability  $\rho^*$  of the system either steadily improves or remains constant as  $|r^*|$  increases. Also, the individual maximum of the two curves  $|\Gamma_{c2}|$ ,  $|\Gamma_{c3}|$  occurs at the tip of the appendages. Therefore,  $\rho^*$ , if not decided by  $|\Gamma_{c1}|$ , reaches a maximum for the location at the tip of the appendages.

In conclusion we can state the following: For the symmetric mode of motion of  $B_F$ , if a single torque actuator is to be used for attitude and shape control the best location for the actuator, from the controllability point of view, is the tip of the appendages regardless of the strength of the actuator, the total time  $T$  allowed for control, and any set of weighting factors one might wish to choose for the normal modal coordinates. (The same conclusion as that for a force actuator.)

Other salient features for the torque actuator for symmetric motion of  $B_F$  are:

- 1) Bad locations in the neighborhood of the zeroes of the influence curves ( $|\Gamma_c|$  curves): Same comments as for the case of a force actuator, in section 5.3, apply.
- 2) Two broad distinct ranges of  $|r^*|$  on  $B_F$ ,  $r_c^* \leq |r^*| \leq 0.21$  and  $|r^*| \geq 0.51$ : Over these ranges controllability of the system either improves or remains constant as  $|r^*|$  increases. Over these ranges the quality of controllability of the  $\theta$  mode

(attitude) remains unaffected by change of location of the actuator, but the quality of controllability of the appendage modes  $\eta_1, \eta_2$  (shape) improves steadily as  $|r^*|$  increases.

- 3) The quality of controllability of the  $\theta$  mode (rigid body mode) is constant for all locations in the system.
- 4) Shape control is not possible for locations on the rigid body  $B_R$ .

These features can be usefully applied as described before (under force actuator) in the search for other suitable locations.

Thus, for the case of the symmetric mode of motion of  $B_F$  we have some valuable information on the quality of locations  $|r^*|$  in the system even with only the knowledge that a torque actuator is to be used.

#### 5.4.3 Antisymmetric Mode of $B_F$

Figure 11 shows the set of  $|\Gamma_c|$  curves for the antisymmetric mode of motion of  $B_F$ . The zeroes of these curves are tabulated in Table 6. For the antisymmetric modes of  $B_F$  the vehicle normal modes are different from the hybrid modes  $\tilde{\eta}(\theta, \eta)$ . But  $\xi_0$  is similar to the rigid body mode  $\theta$  and  $\xi_1, \xi_2$  are similar to the appendage modes  $\eta_1, \eta_2$ . There are three zeroes in all, one for  $|\Gamma_{c2}|$  and two for  $|\Gamma_{c3}|$ . At  $|r^*| = 0$  the system is completely controllable although its degree may be very low. Recall that antisymmetric motion of  $B_F$  produces only rotary motion of  $B_R$  and this can be killed by a torque at  $|r^*| = 0$  thereby eliminating the motion of  $B_F$ . At the locations on  $B_F$  at which  $|\Gamma_{c2}|$ ,

$|\Gamma_{c3}|$  are zero the inertial slopes are constant for the corresponding normal modes (these locations are obtained by putting the inertial angular velocity  $\underline{\Omega}$  of an element on  $B_F$  equal to zero). At these locations the effect of the torque is not felt by the respective modes, and hence they are uncontrollable.

The behavior of the  $|\Gamma_c|$  curves is different compared to those for the symmetric case for locations closer to the rigid body  $B_R$ . Over the rigid body range of locations all the  $|\Gamma_c|$  curves are constant indicating no change in the degree of controllability of the system (or the quality of controllability of any mode) is obtained by moving the actuator (torquer). The system is completely controllable at any location on  $B_R$ . For locations beyond the zero of  $|\Gamma_{c2}|$  ( $|r^*| \approx 0.13$ ) up to the crest of  $|\Gamma_{c3}|$  ( $|r^*| \approx 0.21$ ), and beyond the second zero of  $|\Gamma_{c3}|$  ( $|r^*| \approx 0.51$ ) up to the tip of the appendages the two curves  $|\Gamma_{c2}|$ ,  $|\Gamma_{c3}|$  increase monotonically while the curve  $|\Gamma_{c1}|$  is unaffected. Hence, over the ranges  $0.13 \leq |r^*| \leq 0.21$  and  $|r^*| \geq 0.51$ , the approximate degree of controllability  $\rho^*$  of the system either steadily improves or remains constant as  $|r^*|$  increases. The individual maximum of the two curves  $|\Gamma_{c2}|$ ,  $|\Gamma_{c3}|$  occurs at the tip of the appendages. Therefore,  $\rho^*$ , if not decided by  $|\Gamma_{c1}|$ , reaches a maximum value for the location at the tip of the appendages.

The conclusion here, for the antisymmetric case, is the same as that for the case of symmetric mode of motion of  $B_F$ , in section 5.4.2. The tip of the appendages is the best location for the actuator for attitude and shape control regardless of the actuator strength, total time  $T$  and weighting factors for the normal modes.

Other salient features for the torque actuator for antisymmetric motion of  $B_F$  are:

- 1) Bad locations in the neighborhood of the zeroes of the influence curves ( $|H_c^i|$  curves). Same comments as for all the preceding cases apply.
- 2) Two broad distinct ranges of  $|r^i|$  on  $B_F$ ,  $0.13 \leq |r^i| \leq 0.21$  and  $|r^i| \geq 0.51$ . Over these ranges controllability of the system either improves or remains constant as  $|r^i|$  increases.
- 3) Quality of controllability of all the normal modes is constant for any location on the rigid body  $B_R$  ( $\rho^i$  is constant for the system for these locations).

These features are useful in the search for other suitable locations.

Thus, here also for the case of the antisymmetric mode of motion of  $B_F$ , some valuable information on the quality of locations  $|r^i|$  in the system is obtained with only the knowledge that a torque actuator is to be used.

Summarizing for a single torque actuator control we can say that the tip of the appendages is identified as the best location for attitude and shape control for symmetric and antisymmetric modes of motion of  $B_F$  for this model of a spacecraft which can only deform in the manner described by the two modes of  $B_F$  (obtained from the first and the second cantilever modes of the appendages) considered. This conclusion is independent of the strength of the actuator to be used, the total time  $T$  allowed for control, and the weighting factors prescribed for the modes. Information on regions of poor controllability is useful to check that an actuator is not located close

to any of them. Information on distinct ranges of  $|r^*|$  where improvement in at least shape control is possible is also very useful in the search for alternative locations. For the symmetric case, some understanding of the quality of controllability is gained for the rigid body mode  $\theta$  and the appendage modes  $\eta_1, \eta_2$ . The information for both the symmetric and antisymmetric modes of motion of  $B_F$  taken together provides a better understanding of the system controllability, and identifies certain narrow ranges for locations  $|r^*|$  that will be suitable as alternatives for the torque actuator in order to effectively control both attitude and shape in both types of motion of  $B_F$ .

There is a striking difference between the force actuator control and the torque actuator control. Unlike the case of force actuator control, in torque actuator control there is one mode, a type of rigid body mode (attitude in the case of symmetric motion), whose quality of controllability is unaffected for any location  $|r^*|$  in the system ( $|r_c^*|$  is constant).



### 5.5 Summary for a Single Actuator

For a single actuator, force type or torque type, we gained valuable insight into the quality of locations  $|r^*|$  with respect to the approximate degree of controllability  $\rho^*$  of the system. The major finding was that the tip of the appendages is unconditionally the best location from the point of view of attitude and shape control for either type of actuator. In other words, for our flexible system there exists no other location  $|r^*|$  in the system at which an actuator of either type and of any strength will give a better controllability over any period of time  $T$  for any type of (symmetric or antisymmetric) motion of  $B_F$  under any arbitrary weighting of the modes. This is a very general result. Besides this conclusion, we also observed other significant features in the analysis for a single actuator. One of these was the determination of bad locations for actuators. The other was some distinct ranges of locations  $|r^*|$ , over which the degree of controllability of the system improves steadily as  $|r^*|$  increases, meaning the quality of control of all the modes improves. Over other ranges the quality of shape control definitely improves as  $|r^*|$  increases but the degree of controllability of the system never deteriorates meaning the quality of control of the type of rigid body mode remains constant. These features on the quality of locations  $|r^*|$  are extremely useful in searching for alternative locations that will give reasonably good controllability for the system.

## 5.6 Multiple Actuators

In section 5.3 and 5.4 we analyzed the situation in which a single actuator is used. The analysis provided us with very valuable insight into the behavior of the approximate degree of controllability  $\rho^*$  as the location  $|r^*|$  of the actuator is varied in the system. We will now study the situation in which more than one actuator is used. The aim is merely to extract guide lines for preliminary design of distribution patterns for the actuators.

As in (5.3-1) and (5.4-1) the normal distances  $d_j$  can be written as

$$\begin{aligned} d_1 &= c_1 \left( \sum_{j \in \mathcal{F}} k_{Fj} |\Gamma_{F1}^j| + \sum_{j \in \mathcal{C}} k_{Cj} |\Gamma_{C1}^j| \right) \\ d_2 &= c_2 \left( \sum_{j \in \mathcal{F}} k_{Fj} |\Gamma_{F1}^j| + \sum_{j \in \mathcal{C}} k_{Cj} |\Gamma_{C1}^j| \right) \\ d_3 &= c_3 \left( \sum_{j \in \mathcal{F}} k_{Fj} |\Gamma_{F2}^j| + \sum_{j \in \mathcal{C}} k_{Cj} |\Gamma_{C2}^j| \right) \\ d_4 &= c_4 \left( \sum_{j \in \mathcal{F}} k_{Fj} |\Gamma_{F2}^j| + \sum_{j \in \mathcal{C}} k_{Cj} |\Gamma_{C2}^j| \right) \\ d_5 &= c_5 \left( \sum_{j \in \mathcal{F}} k_{Fj} |\Gamma_{F3}^j| + \sum_{j \in \mathcal{C}} k_{Cj} |\Gamma_{C3}^j| \right) \\ d_6 &= c_6 \left( \sum_{j \in \mathcal{F}} k_{Fj} |\Gamma_{F3}^j| + \sum_{j \in \mathcal{C}} k_{Cj} |\Gamma_{C3}^j| \right) \end{aligned} \quad (5.6-1)$$

where as before  $c_1, c_2, \dots, c_6$  are constants depending on  $T$  and the weighting factors. Let us denote the parenthetical expressions in (5.6-1) as

$\rightarrow e$

$$\Gamma_k^* = \sum_{j \in F} k_{Fj} |\Gamma_{Fk}^j| + \sum_{j \in G} k_{Gj} |\Gamma_{Gk}^j| \quad (5.5-2)$$

Observe that each term of  $\Gamma_k^*$  is nonnegative and represents an independent contribution of an actuator to the normal distances  $d_j$ .

Suppose all  $|\Gamma_{Fk}^j|$  curves increase from  $|\gamma_1^*|$  to  $|\gamma_2^*|$  then all the  $d_j$  increase simultaneously when a force actuator is moved from  $|\gamma_1^*|$  to  $|\gamma_2^*|$ . Continuing in this manner we place one force actuator at the best possible location  $L_1$ . Note that our system is a continuum, but actuators are discrete and occupy only discrete locations. Hence, our analysis may suggest placing actuators at the same locations. From physical considerations we will assume that only one actuator will be placed at any location. This would mean spreading the actuators as a chain. Again from practical considerations we will assume they are separated by some arbitrary distance, and hence discrete locations in a continuous range will be taken in this context. Now, consider the second force actuator and place it at the second best location  $L_2$ , and so on, where these discrete locations are to be interpreted in the context described above. Thus, for each actuator we use the analysis of a single actuator case, which amounts to ranking the locations  $|\gamma^*|$  in descending order of quality for a single force actuator. Now, fill these locations with the force actuators. It is reasonable to distribute the actuators as evenly as possible between the two appendages (there is always a pair of locations, one on each appendage corresponding to any  $|\gamma^*|$ ). One other important observation should be made regarding which force actuator should occupy which

location. From (5.5-2) it can be seen that if we place the strongest actuator at the best location the contribution to the  $d_j$  is further increased. Thus, if over a range of  $|r^*|$ ,  $|r_1^*| \leq |r^*| \leq |r_2^*|$ , all the  $|f_{Fk}|$  curves increase monotonically, and we rank possible locations  $L_1, L_2, \dots, L_p$  in descending order of quality then the actuators  $k_{F1}, k_{F2}, \dots, k_{Fp}$  should be placed at  $L_1, L_2, \dots, L_p$ , respectively, where  $k_{F1} \geq k_{F2} \geq \dots \geq k_{Fp}$ .

Repeat this whole process independently for the torque actuators using the  $|f_{Ck}|$  curves for the case of a single torque actuator. Note that for this case the  $|f_{C1}|$  curve is constant throughout, and hence can be ignored in ranking the locations assuming that we are interested in controlling both attitude and shape. The other two  $|f_{Ck}|$  curves can increase monotonically over certain ranges of  $|r^*|$ . As in the case of force actuators, the torque actuators should be placed such that the strength of an actuator corresponds to the quality of its location in the set of ranked locations (the strongest actuator at the best possible location).

Here we have analyzed the sets of force and torque actuators independently. It is possible that this could ask us to place a force actuator and a torque actuator at the same location. As explained earlier the actuators could be separated by some arbitrary distance so that adjustments can be made to accommodate an actuator from each group.

Since we have used the analysis for a single actuator case to study the multiple actuator case the method outlined here is applicable to all types of motion of the flexible body  $B_F$  that are considered for the single actuator case. To consider both symmetric and antisymmetric

motion of  $B_F$  it is only necessary to take the appropriate ranges of  $|\tau|$  from the single actuator case. In this multiple actuator case we have been concerned only with the ranges of  $|\tau^*|$  over which the  $|\dot{\tau}|$  curves exhibit monotonic behavior. In other ranges of  $|\tau^*|$  the actual behavior of  $\rho^*$  is not apparent, and actual computation of  $\rho^*$  becomes essential including the now important influence of the weighting factors (and the time  $T$ , if nonlinear). The locations close to the tip of the appendages are definitely superior in terms of the resulting degree of controllability for a spacecraft whose flexible body motion is restricted to the two appendage modes considered. Finally, it must be mentioned that the system becomes uncontrollable only if all the force actuators are located at the zeroes of a single  $|\dot{\tau}_{Fk}|$  curve and all the torque actuators are located at the zeroes of the corresponding  $|\dot{\tau}_{ck}|$  curve, which will make a pair of  $d_j$  equal to zero.

In the approach we have outlined for obtaining suitable locations for the case of multiple actuators the following are the main steps:

- 1) Consider the force actuators and torque actuators separately.
- 2) Use the corresponding single actuator analysis independently for each type of actuators. Corresponding to the type of motion of  $B_F$  to be controlled mark out the ranges of  $|\tau^*|$  over which the  $|\dot{\tau}|$  curves exhibit monotonic behavior.
- 3) In these ranges of  $|\tau^*|$  rank locations in descending order of quality.
- 4) Place the force actuators at their corresponding ranked locations in descending order of their strength. Similarly, place the torque actuators at their corresponding ranked

locations. In the case of conflict of locations make adjustments to accommodate the competing actuators in any appropriate manner.

We can consider this approach a logical way to set up preliminary distributions for a set of given actuators. Further modifications and adjustments in the placement of actuators can be carried out based on other practical considerations. If these considerations lead to locations in other regions where the behavior of  $\rho^*$  is not obvious then computation of  $\rho^*$  is necessary with the inclusion of the weighting factors and possibly time  $T$ .

Let us now consider a two actuator case as an example of a multiple actuator case. Following the steps above we can obtain the tip of the appendages as the best location in the system. Since we have two appendages the best distribution for two actuators is one actuator at the tip of each appendage. In a multiple actuator case with more than two actuators two of them can be placed at these tips. The tip being the best location should be the place for the strongest actuator among the force actuators or the torque actuators.

As a remark, observe that the  $|f_c|$  curves for the torque actuators tend to flatten near the tip of the appendages indicating that moving a torque actuator near the tip does not significantly affect the behavior of  $\rho^*$ . This fact may be used in making adjustments in the locations in case of conflicts between force actuators and torque actuators.

## 5.7 Summary

The procedure was described to determine the approximate degree of controllability  $\rho^*$  for a given distribution of actuators, weighting of the modes, and total time  $T$ . Then we proceeded further to analyse the influence curves ( $|\Gamma|$  curves) to gain some insight into the behavior of  $\rho^*$ , while maintaining complete freedom in the choice of the weighting factors and the total time  $T$ . A single force actuator and control and a single torque actuator control were examined for both symmetric and antisymmetric modes of motion of  $B_F$ . We gained some valuable information from this analysis. One conclusion was that the tip of the appendages is the best location in the system for attitude and shape control. This conclusion does not depend on the time  $T$  or the weighting factors. We also identified regions near zones of the curves, which exhibit poor degree of controllability of the system. In all cases, there were two broad distinct ranges of  $|\gamma^*|$  over which  $\rho^*$  behaves monotonically so that locations  $|\gamma^*|$  can be judged qualitatively.

Having obtained some insight into the behavior of \_\_\_\_\_ for a single actuator we next examined the multiple actuator case. We showed that we could essentially treat the multiple actuator case as an extension of the single actuator case provided all the actuators are of the same type. We confined ourselves to some special regions of locations in which the behavior of  $\rho^*$  is simple. A ranking system

was set up for the locations  $|\gamma^*|$ . Also, the relationship of the strength of the actuator to its location was indicated which would give a higher degree of controllability. We extended the analysis to include systems which have multiple actuators of both force and torque types. We showed these could be analyzed independently for each type, then adjustments in the locations of actuators made in case of conflicts.

As an example of multiple actuator systems we considered a two actuator system. This gave the same results as those for a single actuator located on each appendage.

Thus, essentially the analysis of a single actuator case provides valuable information on the quality of locations  $|\gamma^*|$  from the controllability point of view, and this information can be used for multiple actuator systems to obtain in a rational way some reasonably good distribution patterns for the actuators.



## 6. SUMMARY AND CONCLUSIONS

We started with the problem of attitude and shape control of very large flexible spacecraft. In the last four years a large number of potential future spacecraft projects have been identified which require spacecraft of unprecedented size, and hence unprecedented flexibility. The problem is no longer confined to the attitude control of a rigid spacecraft, but both attitude and shape control for the entire vehicle. In order to achieve attitude and shape control actuators have to be distributed over the entire vehicle. How should the locations of the actuators be chosen in order to best control the flexible vehicle? This requires significant advances in the state of the art, and little has appeared in the literature toward this direction.

In this work we started by showing the necessity of a concept of the degree of controllability of a system so that it is possible to compare different distribution patterns of actuators based on this measure. Since the spacecraft dynamic equations can be put into the linear time invariant state space form after linearization about a nominal state, we considered general linear systems for analysis. Then we conducted a search for a meaningful definition of this concept of degree of controllability. Several candidate definitions were scrutinized and found unsuitable, because they did not include all the pertinent factors such as controllability, total time, control

effort, stability, and control objective, which will have a bearing on the degree of controllability. Hence, a definition had to be sought from fundamental physical considerations. The approach led to a meaningful definition for the degree of controllability.

Having formulated this definition we sought for a good approximation so that this is applicable to real problems and numerically manageable, the approximation approaching the true value as the computational effort is increased. The mathematical approach adopted, besides leading us to our goal, showed the system equations in a special form thereby enabling us to derive some relatively simple tests for complete controllability of the system. Thus, we developed an algorithm for computing the degree of controllability of a system.

To apply these concepts to our problem we had to first obtain the state space form for the dynamic equations of spacecraft motion. We analyzed a typical model of a large flexible spacecraft and derived all relevant equations. Then, to carry out a numerical example a specific model of a large flexible spacecraft was examined. The effect of actuator locations on the approximate degree of controllability of the system was analyzed for single and multiple actuator distributions. The single actuator case, for both force type and torque type, yielded valuable information on the quality of locations in the system of actuators with respect to the degree of controllability.

We identified the tip of the appendages as the best

location for a single actuator of a force type or torque type. This is for the model described by two appendage modes. This conclusion is very general in the sense that it is independent of the time  $T$  and the weighting factors associated with the modes. The weighting factors reflect the relative importance in controlling the different modes. There were other salient features of the single actuator analysis. These include: 1) the regions of poor controllability near the zeroes of the influence curves which show the effect of a single actuator location on the degree of controllability, and 2) the regions in which the degree of controllability either steadily improves or never deteriorates (for torque actuator case) as the location of the actuator is changed toward the tip where it reaches its maximum.

For the multiple actuator case, it was shown that the single actuator analysis could be applied for each actuator separately, using the features obtained for the single actuator case, and making adjustments in case of conflicts for the same locations. Thus, the information obtained from the single actuator analysis is applicable to the multiple actuator case, and can be used to set up preliminary distribution patterns for a chosen number and type of actuators. Then the degree of controllability can be evaluated, if necessary, in case some distributions involve locations for which information is not clear from the single actuator analysis (for these, the weighting factors

and time  $T$  may be important). Once this measure of controllability is evaluated it can be decided as to which distribution is of the highest quality (most desirable) from the controllability point of view.

In this work the following can be considered as the main contributions:

- 1) The concept of the degree of controllability-- a meaningful definition for all linear time invariant systems (actually the definition is applicable to nonlinear systems also).

- 2) An algorithm for a reasonable approximation for the degree of controllability based on an approximate bound for the recovery region. This approximation satisfies the property that it is zero if and only if the true degree of controllability is zero.

This algorithm is relatively simple from the numerical point of view. Using this approximate measure various distributions of actuators can be ranked in descending order of their desirability in a practical application.

- 3) A logical and a very useful approach for preliminary distribution patterns for the actuators so that a designer need not indulge in a blind search for some distributions to be studied.

- 4) Relatively simple controllability tests for all linear time invariant systems. The minimum number of actuators required for complete controllability of a system is a by-product of these tests.

As a final remark, since the analysis for the degree of controllability is based on linear time invariant systems it can be applied to problems other than the spacecraft actuator distributions. This will involve identifying a parameter, analogous to actuator location, which will affect the degree of controllability of a given system. Of course, the equilibrium state must also be identified.

Table 1.  
Appendage Modal Constants

Qty.	Cantilever mode number j	
	First mode (j=1)	Second mode (j=2)
$\beta_j L$	1.375	4.594
$\omega_j$	$\sqrt{(EI)/\rho A L^4} (\beta_j L)^2$	$\sqrt{(EI)/\rho A L^4} (\beta_j L)^2$
$\gamma_j$	0.734	1.0135
$x_{0j}^A$	-----	0.7334
$a_j$	0.7329771	0.43351571
$e_j$	0.55381020	0.00250375
$q_j$	0.99933321	1.00032273

Table 2.  
Data for Specific Model of a Large Flexible  
Spacecraft

No.	Quantity	Symbol	Value
1	Radius of the central hub, $a_h$	$a_{hub}$	2.0 ft.
2	Length of either appendage		50.0 ft.
3	Linear mass density of either appendage	$c_a$	$1.49 \times 10^{-4}$ slug/ft.
4	Normal distance from C to the appendage axis	$r_h$	0.0 ft.
5	Stiffness of either appendage Inertia of the central hub about point C	$(EI)$ $I_h$	$15.5 \text{ lb-ft}^2$ 117.0 slug ft <sup>2</sup>

Approved for Release by NSA on 08-24-2013 pursuant to E.O. 13526

Table 3.  
Input System Data

Quantity	Value
$r_e^4$	0.03333333
$\frac{c_s z}{m_y l}$	0.0
$\frac{m_s}{m_F}$	1035.7452
$\frac{m_s l^2}{I_s}$	1119.2377
$\frac{EI}{c_p l^4}$	$2.66357 \times 10^{-3} \text{ sec}^{-2}$
$\omega_1$	$0.131444 \text{ sec}^{-1}$
$\omega_2$	$1.137172 \text{ sec}^{-1}$

Table 4.  
System Constants

Quantity	Mode of motion of $R_F$	
	Symmetric	Antisymmetric
$\nu_1$	55.912573	55.911275
$\nu_2$	55.936593	55.942302
$\lambda_1$	0.0	0.0
$\lambda_2$	$3.294051 \times 10^{-3}$	0.051792
$\lambda_3$	1.293785	1.117019



Table 5.

Locations of Zeroes of Influence Curves for Force Actuators

Curve	Mode of Motion of $B_F$				
	Symmetric		Antisymmetric		
	Location $ r $		Location $ r $		
$ r_{F1} $	0.0	--	0.0	----	----
$ r_{F2} $	0.0477	--	0.0	0.255	----
$ r_{F3} $	0.0376	0.8166	0.0	0.0545	0.815

Table 6.

Locations of Zeroes of Influence Curves for Torque Actuators

Curve	Mode of Motion of $B_F$			
	Symmetric		Antisymmetric	
	Location $ r $		Location $ r $	
$ r_{c1} $	----	----	-----	-----
$ r_{c2} $	$0 \leq  r^*  \leq r_e^*$ <sup>†</sup>	-----	0.127	-----
$ r_{c3} $	$0 \leq  r^*  \leq r_e^*$	c.505	0.0375	0.505

<sup>†</sup> This is over the entire range of  $B_R$ .

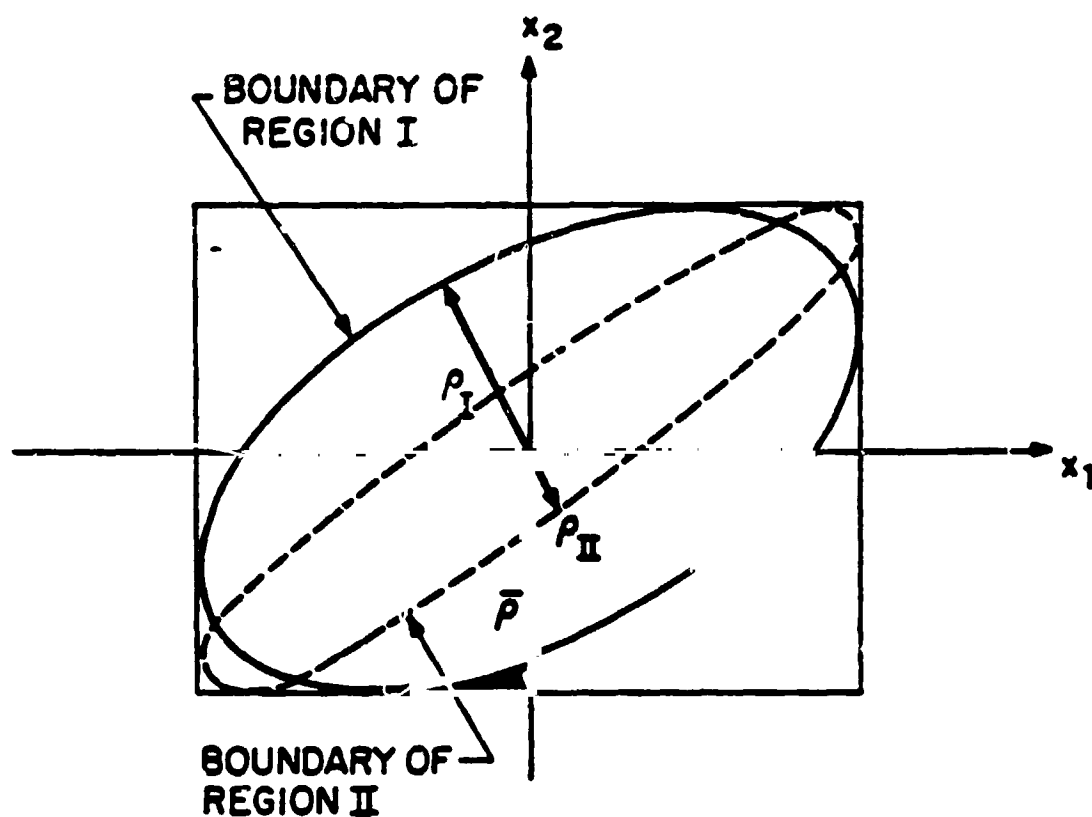


Fig. 1. The recovery region and its rectangular approximation

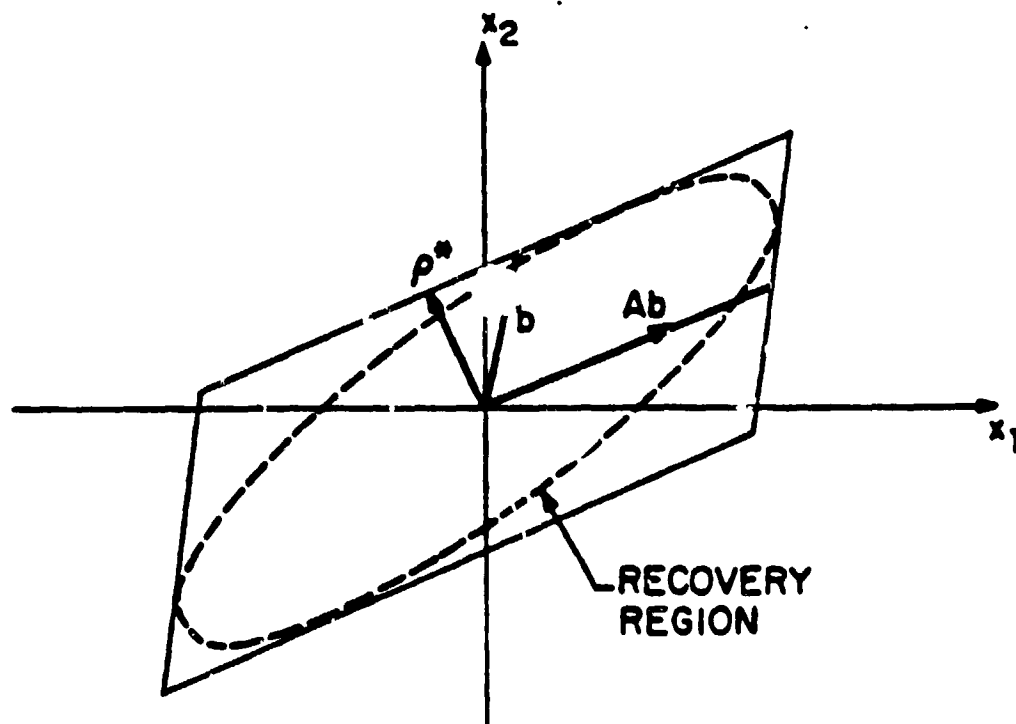


Fig. 2. Parallelogram bound on the recovery region

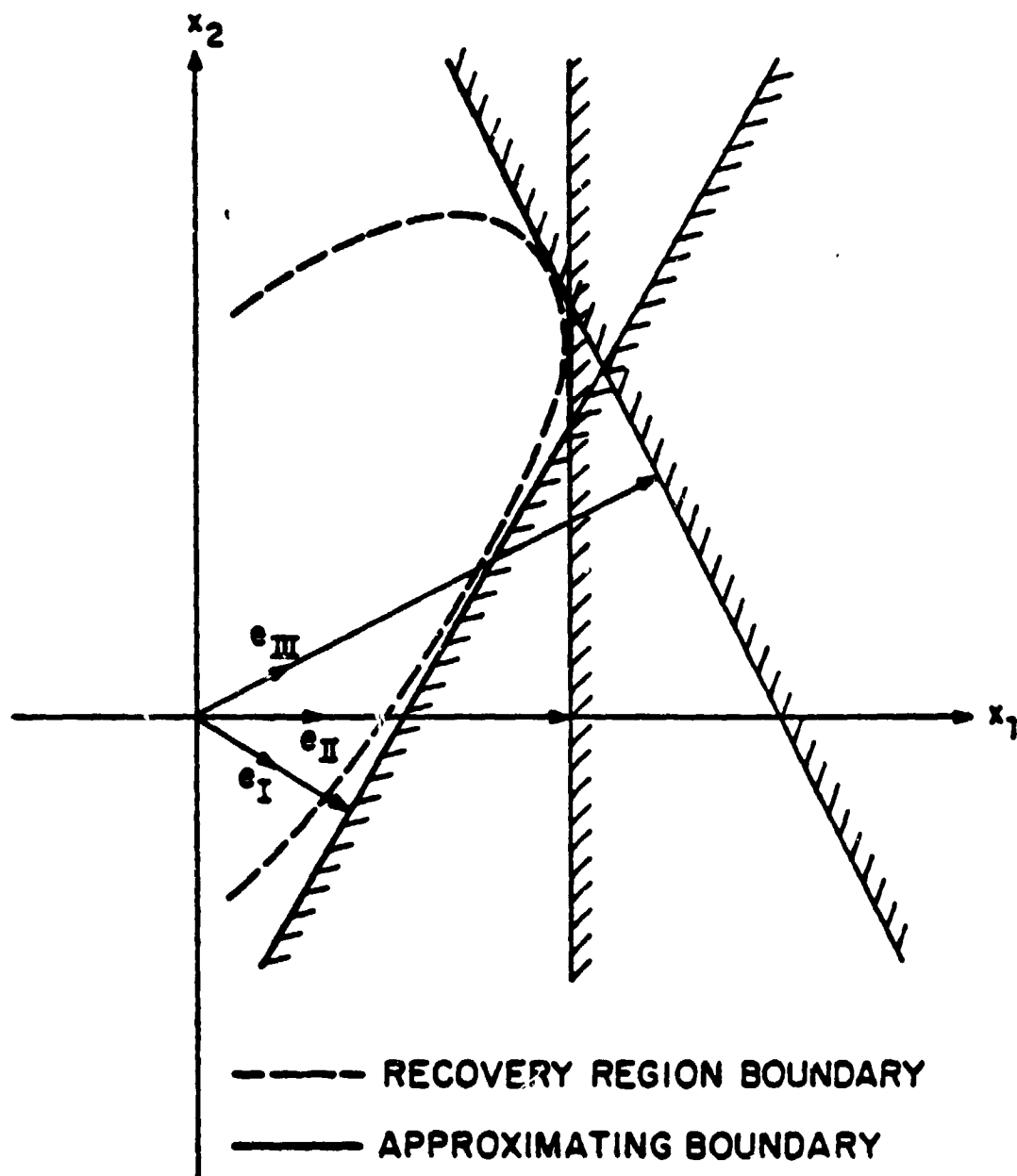


Fig. 3. Improving the approximation to the recovery region

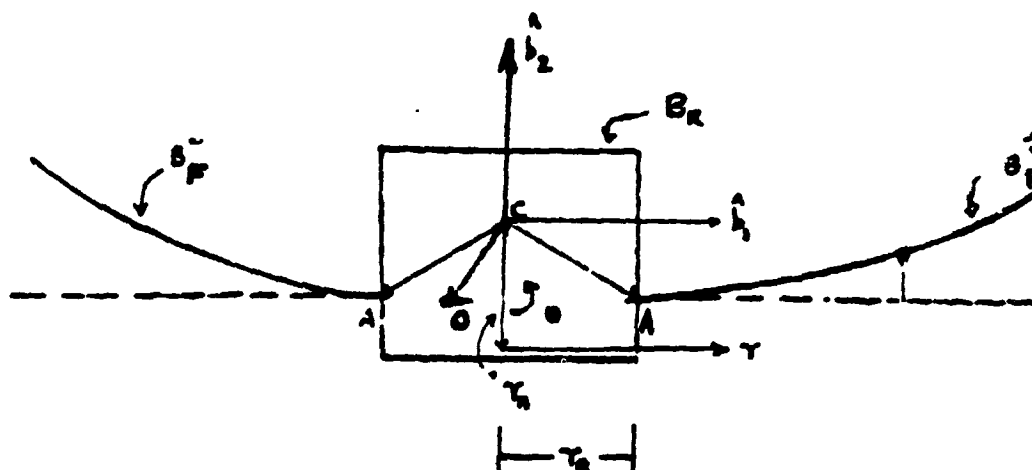


Fig. 4 A Typical Planar Motion Model of a Large Flexible Spacecraft

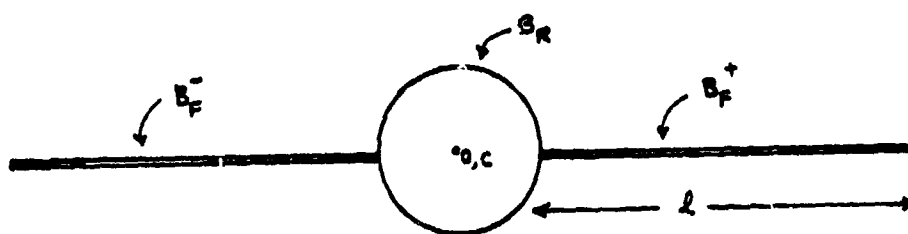


Fig. 5 A Specific Model of a Very Large Flexible Spacecraft

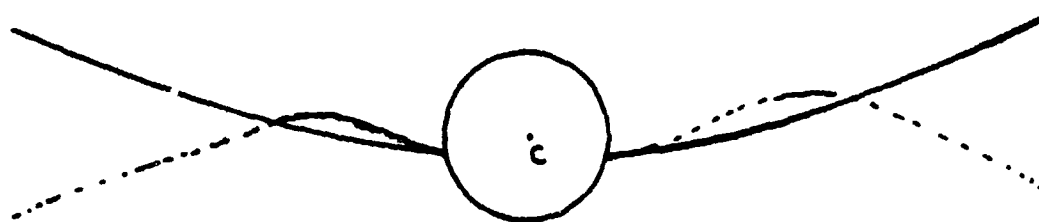


Fig. 6 Spacecraft in Symmetric Mode of Motion of Flexible Appendages

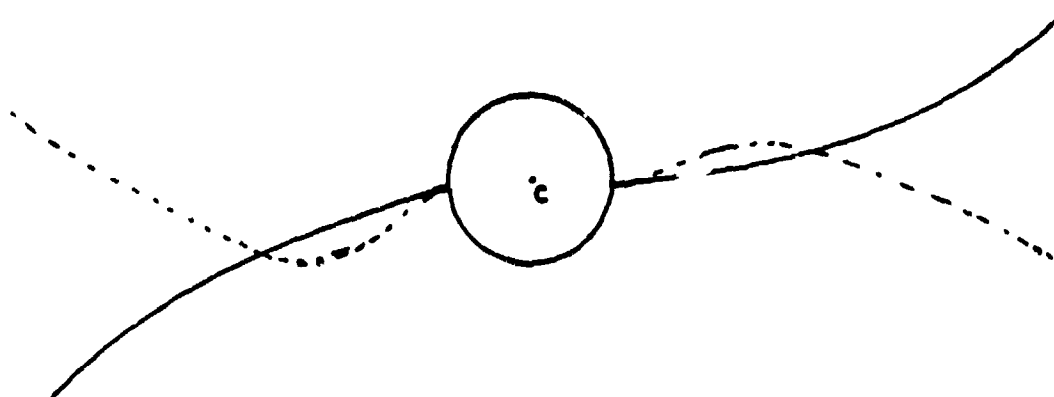


Fig. 7 Spacecraft in Antisymmetric Mode of Motion of Flexible Appendages

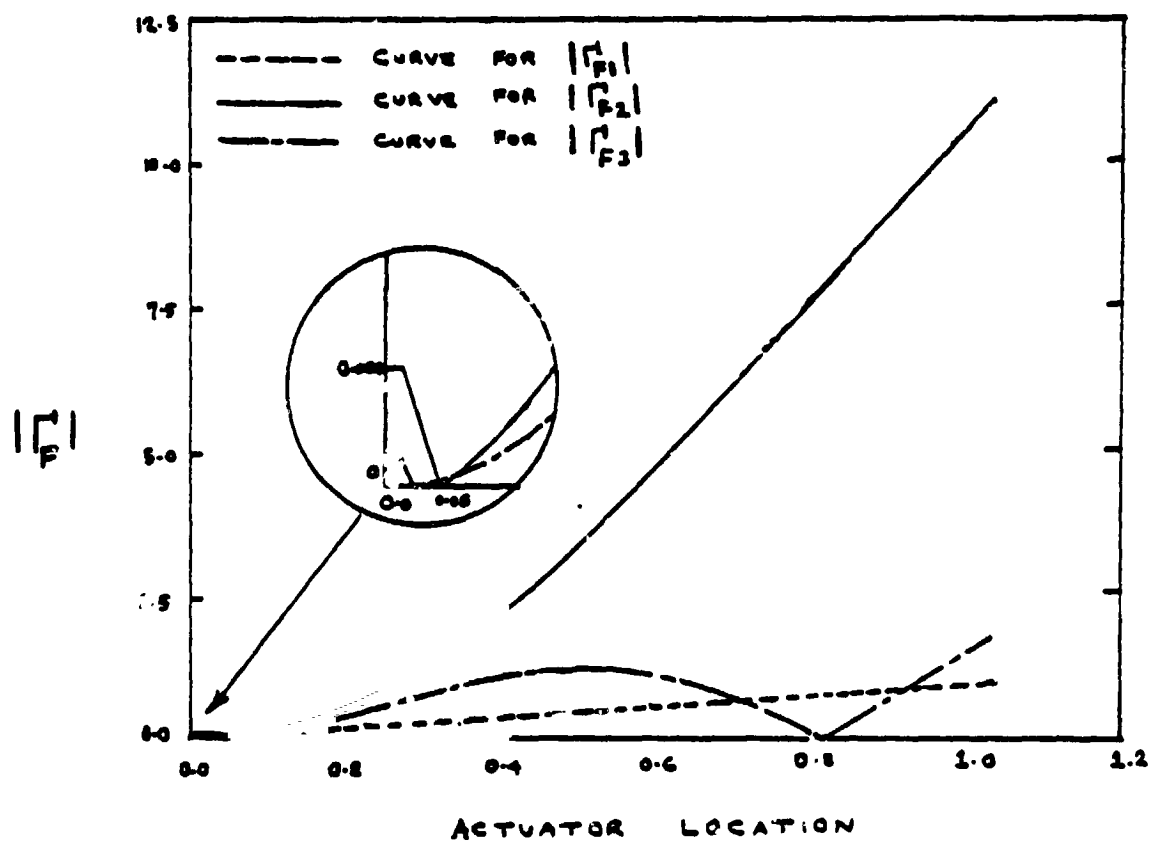


Fig. 8 Influence Curves for Force Actuators  
for Spacecraft in Symmetric Motion

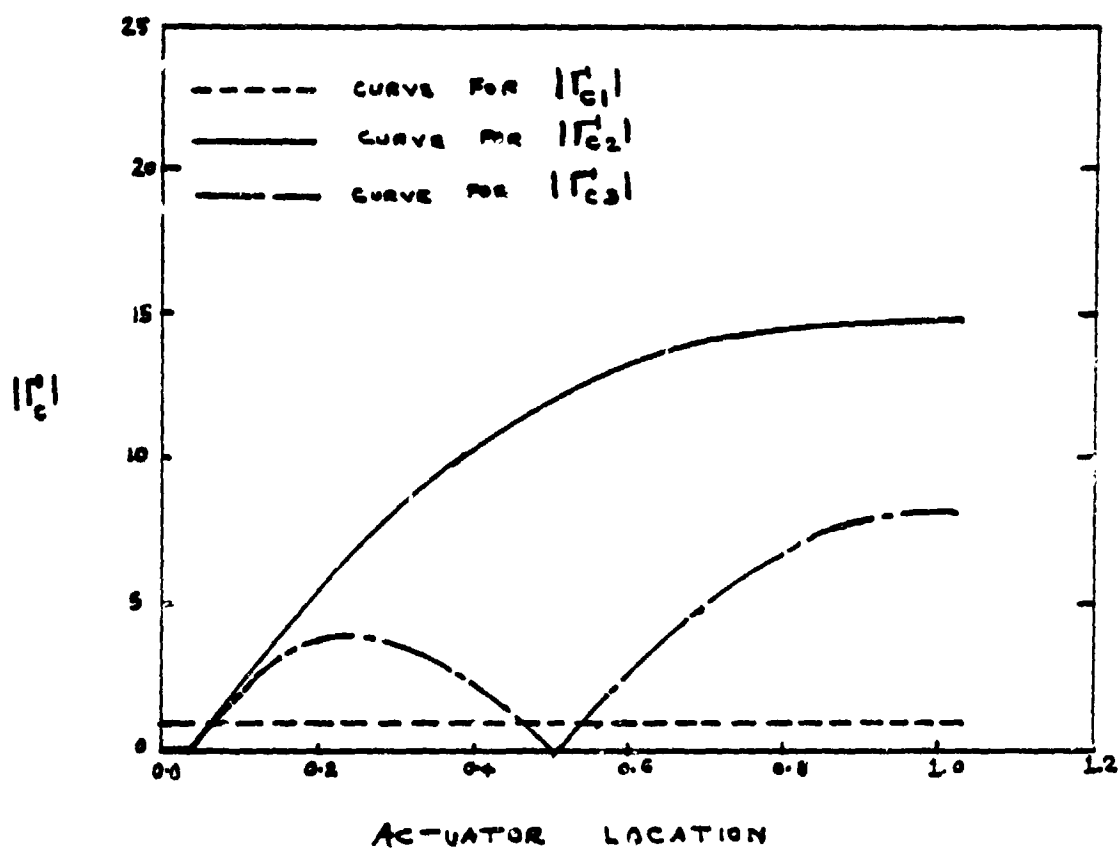


Fig. 9 Influence Curves for Torque Actuators  
for Spacecraft in Rotational Motion

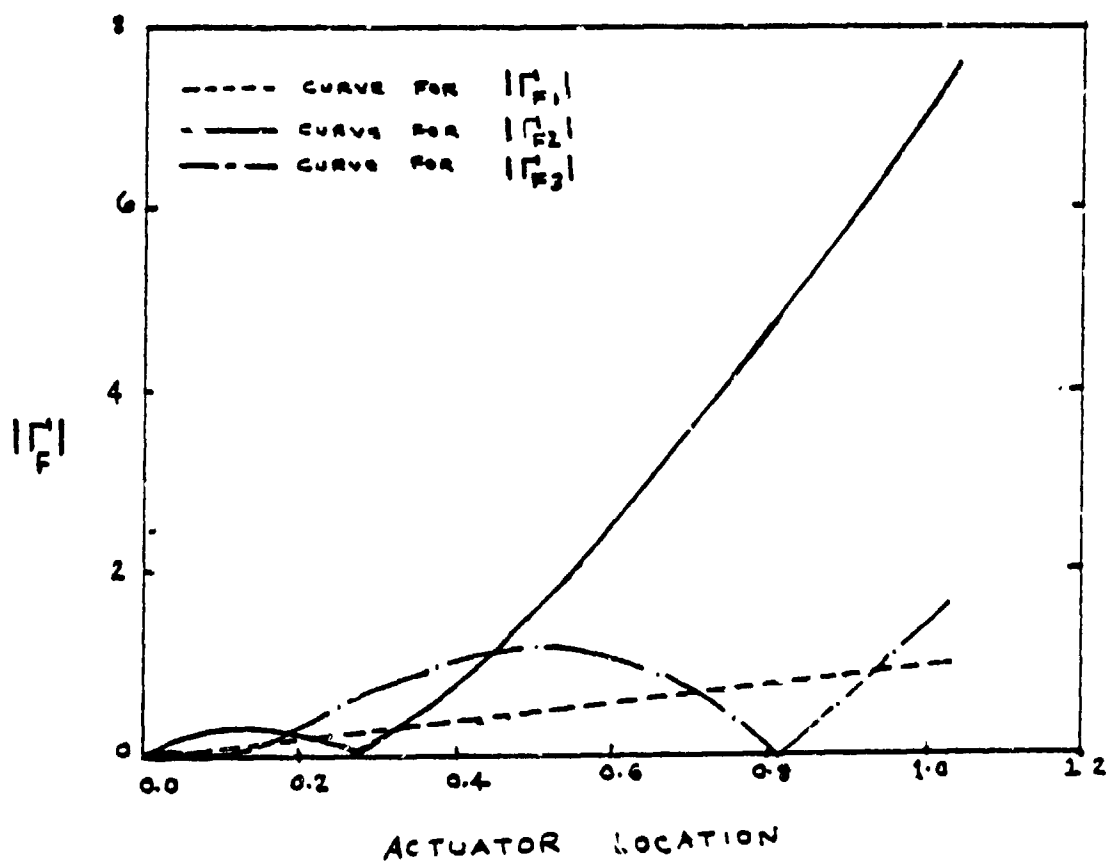


Fig. 10 Influence Curves for Force Actuators  
for Spacecraft in Antisymmetric Motion



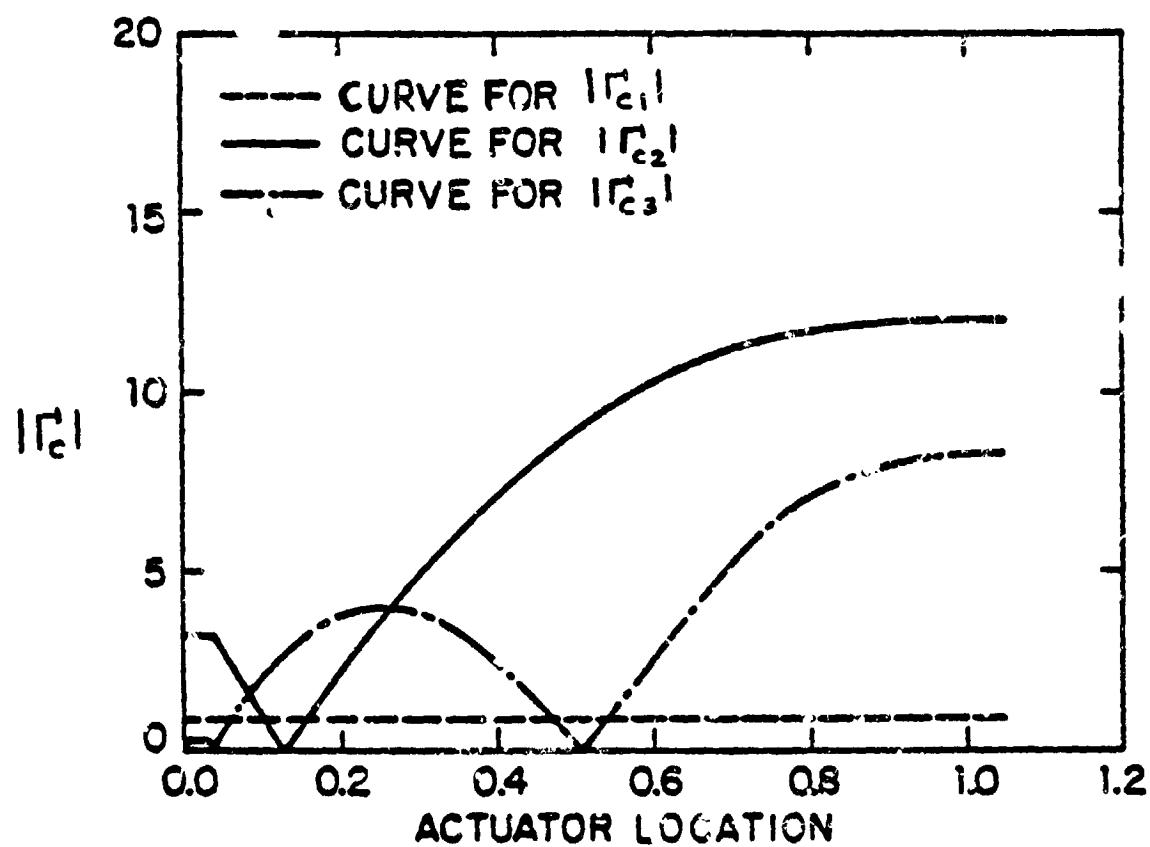


Fig. 11 Influence Curves for Torque Actuators  
for Spacecraft in Antisymmetric Motion

## 7. REFERENCES

1. "The New Generation of Dynamic Interaction Problems", P. W. Likins, AAS Annual Rocky Mountain Guidance and Control Conference, March, 1978.
2. "State Space Analysis of Control Systems", K. Ogata, Prentice-Hall, Inc., 1967, pp. 301-307, pp. 378-381.
3. "Optimal Controls", M. Athans and P.L. Falb, McGraw-Hill, New York, 1966.
4. "Applied Linear Algebra", B. Noble, Prentice-Hall, Inc. 1969, pp. 335-338.
5. "Dynamics and Control of Flexible Space Vehicles", P. W. Likins, Technical Report 32-1329. Revision 1, NASA Jet Propulsion Laboratory, Jan. 1970.
6. "Applications of the Theory of Matrices", F.R. Gantmacher, Interscience Publishers, N.Y. 1959.
7. "Elements of Vibration Analysis", L. Meirovitch, McGraw-Hill, 1975.

## 8. APPENDICES

## A. Matrix Algebra

(i) By repeated postmultiplication of  $P^{-1}AP=J$  in (2.4-1) by  $P^{-1}A$  on the left side and  $JP^{-1}$  on the right side it follows directly that

$$P^{-1}A^\alpha = J^\alpha P^{-1} \quad (A-1)$$

$$\alpha = 1, 2, \dots$$

(ii) The exponential of any matrix  $-Rt$  can be written as

$$e^{-Rt} = E - Rt + \frac{R^2 t^2}{2!} - \frac{R^3 t^3}{3!} + \dots \quad (A-2)$$

Suppose  $R=A$ , then premultiply both sides by  $P^{-1}$  to obtain

$$\begin{aligned} P^{-1}e^{-At} &= P^{-1} - P^{-1}At + P^{-1}A^2 \frac{t^2}{2!} - P^{-1}A^3 \frac{t^3}{3!} + \dots \\ &= \left[ E - Jt + J^2 \frac{t^2}{2!} - J^3 \frac{t^3}{3!} + \dots \right] P^{-1} \\ &= e^{-Jt} P^{-1} \end{aligned} \quad (A-3)$$

where (A-1) has been used for each term on the right side to obtain the series in  $J$  in the paranthesis (this is obtained from (A-2) by substituting  $R=J$ ).

(iii) From (2.4-1) we can immediately write

$$J^\alpha = \text{diag} [J_1^\alpha, J_2^\alpha, \dots, J_r^\alpha]$$

$$\alpha = 0, 1, 2, \dots$$

where  $J^0 = E$ , the unit matrix. Substitution of this on the right side of (A-2) for  $R=J$  for all powers of  $\alpha$ , leads to

$$e^{-Jt} = \text{diag} [e^{-J_1 t}, e^{-J_2 t}, \dots, e^{-J_r t}] \quad (\text{A-4})$$

where  $J_k$  are the Jordan blocks of order  $\nu_k$ .

(iv) The matrix  $J_k$  can be written as

$$J_k = \lambda_k E_k + \hat{N}_k$$

where  $E_k$  is the unit matrix of order  $\nu_k$ , and  $\hat{N}_k$  is a nilpotent matrix of index  $\nu_k$  as shown below.

$$\hat{N}_k = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{A-5})$$

$$\hat{N}_k^{\nu_k} = [0] \quad (\text{A-6})$$

Because  $\lambda_k E_k$  and  $\hat{N}_k$  are commutative  $e^{-J_k t}$  can be written as

$$e^{-J_k t} = e^{-(\lambda_k E_k + \hat{N}_k)t} = e^{-\lambda_k E_k t} e^{-\hat{N}_k t} \quad (\text{A-7})$$

Expanding  $e^{-\lambda_k E_k t}$  by using (A-2) ( $R = \lambda_k E_k$ ) we obtain the result

$$e^{-\lambda_k E_k t} = e^{-\lambda_k t} E_k \quad (A-8)$$

Substituting this in (A-7), we have

$$e^{-J_k t} = e^{-\lambda_k t} e^{-\hat{N}_k t} \quad (A-9)$$

Again, by substituting  $R = \hat{N}_k$  in (A-2) and using (A-6) we get

$$e^{-\hat{N}_k t} = E_k - \hat{N}_k t + \frac{\hat{N}_k^2 t^2}{2!} - \frac{\hat{N}_k^3 t^3}{3!} + \dots + \hat{N}_k^{j_k-1} \frac{(-t)^{j_k-1}}{(j_k-1)!} \quad (A-10)$$

The powers of  $\hat{N}_k$  are easily evaluated by shifting the superdiagonal in (A-5). Hence, by expansion of  $\hat{N}_k^\alpha$ ,  $\alpha = 1, 2, 3, \dots$

...,  $j_k-1$ , in (A-10), and adding up all the terms we obtain for  $e^{-\hat{N}_k t}$  the following form:

$$e^{-\hat{N}_k t} = \begin{bmatrix} 1 & -t & \frac{t^2}{2!} & \frac{(-t)^3}{3!} & \dots & \frac{(-t)^{j_k-1}}{(j_k-1)!} \\ 0 & 1 & -t & \frac{t^2}{2!} & \dots & \frac{(-t)^{j_k-2}}{(j_k-2)!} \\ 0 & 0 & 1 & -t & \dots & \frac{(-t)^{j_k-3}}{(j_k-3)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (A-11)$$

## APPENDIX B

### A Definition of the Degree of Controllability--A Criterion for Actuator Placement<sup>1</sup>

C. N. Viswanathan  
Graduate Research Assistant  
Columbia University  
New York, N.Y. 10027

R. W. Longman  
Professor of Mechanical Engineering  
Columbia University  
New York, N.Y. 10027

P. W. Likins  
Professor and Dean  
School of Engineering and Applied Science  
Columbia University  
New York, N.Y. 10027

### ABSTRACT

The unsolved problem of how to control the attitude and shape of future very large flexible satellite structures represents a challenging problem for modern control theory. One aspect of this problem is the question of how to choose the number and locations throughout the spacecraft of the control system actuators. Starting from basic physical considerations, this paper develops a concept of the degree of controllability of a control system, and then develops numerical methods to generate approximate values of the degree of controllability for any spacecraft. These results offer the control system designer a tool which allows him to rank the effectiveness of alternative actuator distributions, and hence to choose the actuator locations on a rational basis. The degree of controllability is shown to take a particularly simple form when the satellite dynamics equations are in modal form. Examples are provided to illustrate the use of the concept on a simple flexible spacecraft.

### INTRODUCTION

In the last few years a large number of potential future satellite projects have been identified which require spacecraft of unprecedented size, and hence unprecedented flexibility. The attitude control problem for such a spacecraft is best characterized as simultaneous pointing control and shape control of the vehicle. In order to achieve shape control, or equivalently control of the various modes of oscillation of the flexible structure, it will be necessary to distribute actuators throughout the

<sup>1</sup>This research was supported by NASA Contract NAS 8-32212 with the NASA Marshall Space Flight Center.

vehicle. How should the number and locations of actuators be chosen in order to best control the flexible spacecraft?

This problem has been recognized for some time, but to date little has appeared in the literature that would help guide the control system designer in placing the actuators. Most of the known results identify the minimum number of actuators needed for a given set of modes to be controlled, and identify certain specific actuator locations which cannot be used because they result in an uncontrollable system.

The concept of controllability in modern control theory is a binary concept, either a system is controllable or it is uncontrollable. Starting from a set of actuator locations which produce an uncontrollable system, but for which the number of actuators is sufficient to produce controllability, it will usually be the case that moving one of the actuators by a distance  $\epsilon > 0$  can produce a controllable system, no matter how small the  $\epsilon$ . One expects that for a small  $\epsilon$ , even though technically the system is controllable, in some sense it will not be very controllable. A precise definition of this concept would prove useful for actuator placement.

It is the purpose of this paper to generate, starting from basic physical considerations, a definition of the degree of controllability. The definition obtained is certainly not the only possible definition, but it does have the advantage over a definition based on singular value decomposition that the physical reality of actuator saturation limitations is included in a fundamental way, and that time limitations on accomplishing one's control objective can be included.

The definition is then applied to the actuator placement problem for flexible spacecraft. With this tool the control system designer can rank the desirability of various candidate actuator distributions, and thus he would have a rational way of picking which distribution to use.

#### DEFINITION OF THE DEGREE OF CONTROLLABILITY

Let us consider any general linear time invariant system in state variable form

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t) \quad (1)$$

where  $x^* \in \mathbb{R}^n$  and  $u^* \in \mathbb{R}^m$ . It should be noted that although we focus our attention on this system, the degree of controllability definition which we adopt is also applicable to more general systems of the form  $\dot{x}^*(t) = f(x^*, u^*, t)$  having a solution  $x^*(t) \equiv 0$  ( $f(0, 0, t) = 0$ ).

It is instructive to discuss some of the candidate definitions of the degree of controllability which were considered and discarded--the process of starting with a blind attempt at a definition and progressing to a well formulated concept highlights the characteristics that a workable definition must have. It is tempting to try to connect the degree of controllability to properties of the standard controllability matrix  $Q = [B \ AB \ \dots \ A^{n-1}B]$ , and define degree of controllability as the square root of the minimum eigenvalue of  $QQ^T$ . Four apparent difficulties with this definition must somehow be handled before the definition becomes viable. These difficulties are as follows: 1) The degree of controllability is affected by a transformation of coordinates (since the eigenvalues of  $QQ^T$  are not invariant under changes in state variable representation). 2) This candidate definition satisfies the basic requirement that



the degree of controllability is zero when the system is uncontrollable, but it is not immediately clear what other physical meaning can be attached to the size of the eigenvalues of  $QQ^T$ . 3) The candidate definition does not involve a dependence on the amount of time  $T$  allotted to accomplish the control task. It can be much easier to control the system state in some directions in the state space at one time than at another time, so the degree of controllability should depend on  $T$ . 4) It is not clear that the amount of control effort needed to accomplish the control task is reflected in this definition. In the satellite described above where one actuator has been moved by a small amount  $\epsilon$  to produce controllability, one expects the "weak controllability" of the system to be manifested in the need for very large control actions to accomplish certain small changes in the state, and hence the control effort required is of fundamental importance in making a definition.

It is clear that some type of limitation or standardization of the control effort must be included in the definition. Consider a standardization which restricts the control to a unit impulse, and consider systems with  $A$  in diagonal form and with  $u^*$  a scalar. For distinct eigenvalues the system is controllable if none of the elements  $b_i$  of the column matrix  $B$  are zero. Furthermore, these components indicate how far a unit impulse control will move each state component instantaneously, so one might suggest the  $\min |b_i|$  as a degree of controllability. Here we are trying to generalize a second standard test for controllability to obtain a degree of controllability definition. Among the apparent difficulties with this candidate definition is the fact that the control actions are so restricted that the components of the state cannot be affected independently. The control of all states by a single control  $u^*$  relies on the differences in the dynamic behaviors of the states.

Both of these candidate definitions have difficulties; they do not appear to include the effects of all pertinent variables. Hence, it will be necessary to build the definition from more fundamental considerations. Ironically, when this is completed and interpreted properly, in certain special cases the degree of controllability definition will essentially reduce to the second candidate definition above (and by employing a different approach involving singular value decomposition of matrices something of the general form of the first candidate definition can result).

It is now evident that the definition of the degree of controllability, besides being in some sense a measure of how easy it is for the controller to control the system, must in some way handle four things:

- 1) It must have the property that the degree of controllability is zero when the system is uncontrollable.
- 2) It must somehow consider dependence on total time  $T$ .
- 3) It must standardize the control effort in some way.
- 4) The control objective must be restricted.

Concerning the last point, certainly different control objectives should influence the choice of the control system design, and hence the degree of controllability of a candidate design should be keyed to the objective involved. In a large class of problems (regulator problems), the equilibrium solution  $x^*=0$  to equation (1) is of primary importance, and the control objective is to return  $x^*$  to zero after a disturbance. Since this is the most common attitude and shape control problem for flexible spacecraft, we will restrict ourselves to this objective. A companion

paper [1] develops the concept of the degree of controllability for various satellite slew maneuver objectives. Concerning the standardization of the control effort we will require that the control components satisfy  $|u_i| \leq 1$  for  $i=1,2,3,\dots,m$ , which represents realistic physical limitations of the actuator capabilities. Note that the use of one as the bound for all control components implies normalizing each component of  $u^*$  to produce a new control vector  $u$ , and adjusting the  $B$  matrix to produce a new matrix  $B$ .

Controllability requires the existence of a control function which can transfer any initial state to any final state in finite time. With our more limited control objective, the degree of controllability should be related to the volume of initial system states (or states resulting from disturbances) which can be returned to the desired state  $x^*=0$  in time  $T$  using the bounded controls. Consider the nature of this volume in more detail. In an uncontrollable system there will be at least one direction in the state space for which initial conditions in this direction cannot be returned to the origin, and the volume will lose one or more dimensions. For a controllable system whose parameters are such that it is nearly uncontrollable, only initial conditions very close to  $x^*=0$  along the above mentioned direction could be returned to the origin in time  $T$  using the bounded controls. Hence, we will generate a definition of the degree of controllability based on the minimum distance from the origin to a normalized state that cannot be brought to the origin in time  $T$ . More loosely it is the minimum disturbance from which the system cannot recover in time  $T$ .

The coordinates of a state space will very rarely all have the same physical units, and hence it is clear that comparing distances in the state space will require that each coordinate must be made unitless by normalization. How should one choose the normalization to use? Recognize that when comparing two controller designs for controlling the same dynamic system, the needed minimum distance for each design will usually correspond to a different direction in state space. Hence, ranking of the degree of controllability of the two systems will depend on comparison of distances in different directions, and this implies that we must be equally interested in controlling deviations of the state from  $x^*=0$  in all directions in the state space. In order to accomplish this the control system designer must specify  $n-1$  numbers which represent his degree of interest in controlling each component of the state. This could be done, for example, by determining the deviations of  $x_1^*, x_2^*, \dots, x_{n-1}^*$  which would be considered of equal importance to a deviation of  $x_n^*=1$ . The reciprocals of these numbers would then be used to produce normalization factors for each of the coordinates of the state space giving a new state vector  $x$ . The system equations expressed in terms of the normalized state  $x$  and normalized control  $u$  are then written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ |u_i| &\leq 1 \quad i=1,2,3,\dots,m \end{aligned} \quad (2)$$

Just as in optimal control theory where the control system designer must be specific about his goal by specifying a cost functional, in order to define the degree of controllability, it is necessary to be fully specific not only about the objective of keeping  $x=0$ , but also about the relative importance of keeping each component of  $x$  near zero.

Relative to the normalized system (2) we are now ready to make the

following definitions:

Definition 1: The recovery region for time T for normalized system (2) is the set

$$R = \{ x(0) \mid \exists u(t), t \in [0, T], |u_i(t)| \leq 1 \text{ for } i = 1, 2, \dots, m, x(T) = 0 \}$$

Definition 2: The degree of controllability in time T of the  $x=0$  solution of normalized system (2) is defined as

$$\rho = \inf \|x(0)\| \quad \forall x(0) \notin R$$

where  $\|\cdot\|$  represents the Euclidian norm.

Thus, the recovery region identifies all of the initial conditions (or disturbed states) which can be returned to the origin in time T using the bounded controls. And the degree of controllability is a scalar measure of the size of the region, where the scalar is chosen as the shortest distance from the origin to an initial state which cannot be returned to the origin in time T.

The degree of controllability, as defined, is keyed to the state vector  $x$  employed. No transformations of coordinates can be allowed once the normalization has been specified (unless the norm used in the definition is adjusted to compensate for the resulting distortion of the state space).

Note the following property of the recovery region:

Remark: The recovery region  $R$  for system (2) is the same as the set of reachable states for time T for the system

$$\begin{aligned} \dot{x}(t) &= -Ax(t) - Bu(t) \quad t \in [0, T] \\ |u_i| &\leq 1 \quad i=1, 2, 3, \dots, m \end{aligned}$$

starting from  $x(0)=0$ .

It should be pointed out that although Definition 2 incorporates all the properties which were identified as necessary in the definition of the degree of controllability, it is not necessarily unique in doing so. For example, a standardization of the control effort in terms of energy can also be employed, but the inequality saturation constraints on the controls used here represents the more realistic situation.

An example section is provided in this paper in order to illustrate how the degree of controllability behaves as a function of actuator placement in a simple flexible spacecraft, and to demonstrate that the definition behaves according to our limited intuitive notion of the degree of controllability.

### CONCEPTS FOR APPROXIMATING THE RECOVERY REGION

In order to make the definition of the degree of controllability useful, it is necessary to develop a simple algorithm to generate at least an approximation to the distance  $\rho$ . This necessitates approximating the recovery region  $R$ .

Note that

$$x(T) = \Phi(T, 0)x(0) + \Phi(T, 0) \int_0^T \Phi(0, t)Bu(t)dt \quad (3)$$

where  $\Phi$  is the state transition matrix for (2). The distance moved during time T is  $\xi = x(0) - x(T)$ , and we are concerned with sending the system to the origin so that  $x(T)=0$ . Then the initial state  $\xi$  which reaches the origin in time T using control  $u(t)$  is given by

$$-\xi = \int_0^T \Phi(0, t)Bu(t)dt \quad (4)$$

By the Caley-Hamilton theorem the state transition matrix can be written as

$$\Phi(0, t) = e^{-At} = \sum_{\alpha=0}^{n-1} \psi_{\alpha}(t)A^{\alpha} \quad (5)$$

where the  $\psi_\alpha$  are scalar functions of time. Partition the B matrix into column matrices  $b_j$ , and define the following matrices

$$B = [b_1 \ b_2 \ b_3 \ \dots \ b_m] \quad (6)$$

$$\psi^T = [\psi_0 \ \psi_1 \ \psi_2 \ \dots \ \psi_{n-1}] \quad (7)$$

$$Q = [B \ AB \ A^2B \ \dots \ A^{n-1}B] \quad (8)$$

$$Q_\beta = [b_\beta \ Ab_\beta \ A^2b_\beta \ \dots \ A^{n-1}b_\beta] \quad (9)$$

Then  $\xi$  can be represented in the following alternative forms

$$-\xi = \sum_{\alpha=0}^{n-1} \sum_{\beta=1}^m \left\{ \int_0^T \psi_\alpha(t) u_\beta(t) dt \right\} A^\alpha b_\beta \quad (10)$$

$$= \sum_{\beta=1}^m \int_0^T [\psi_0 b_\beta + \psi_1 A b_\beta + \dots + \psi_{n-1} A^{n-1} b_\beta] u_\beta dt \quad (11)$$

$$= \sum_{\beta=1}^m \int_0^T [Q_\beta \psi] u_\beta dt \quad (12)$$

For the purposes of illustrating certain concepts, let us restrict ourselves to the case of a scalar control so that the summations over  $\beta$  as well as the  $\beta$  subscripts in the above can be dropped, and B is a column matrix  $b$ . Also let  $n=2$  for simplicity. Suppose the recovery region is as shown by Region I in Fig. 1. The maximum  $x_1$  component of any state in the recovery region is obtained by using the control  $u$  equal to minus the signum function of the first component of the vector  $[Q\psi]$  in (12), since this maximizes the  $x_1$  component of the integrand at each time  $t$ . The right hand side (and left hand side) of the rectangle enclosing this recovery region in Fig. 1 can thus be found by integrating the first component in (12) using this control. If desired the point at which the recovery region touches this side is obtained by integrating the second component of (12) using this control. The top and bottom of the rectangle are found similarly.

The rectangle obtained in this manner might be considered an approximation to the recovery region, and then the shortest distance from the origin to one of the sides might be considered an approximation,  $\bar{\rho}$ , to the degree of controllability,  $\rho=\rho_I$ . Note that this necessarily produces a  $\bar{\rho}$  which is an upper bound for the degree of controllability. In some cases this approximation is a tight one, but often it is not. Suppose the recovery region was Region II of Fig. 1. This corresponds to a system which has a much poorer degree of controllability,  $\rho=\rho_{II}$ , yet the approximation  $\bar{\rho}$  remains the same. In fact, suppose that  $\rho_{II} \rightarrow 0$  in such a way that Region II degenerates to a line forming a diagonal of the rectangle in Fig. 1. Then the system is an uncontrollable system, but the approximation  $\bar{\rho}$  still predicts a good degree of controllability. Hence, this approximation must be rejected.

For the case of a scalar control being considered, this shortcoming can be eliminated by using  $-\xi$  as expressed in (10) and maximizing components along  $A^\alpha b$ . The control

$$u(t) = -\text{sgn}[\psi_\alpha(t)] \quad (13)$$

extremizes the coefficient of the vector  $A^\alpha b$  in (3). It will simultaneously produce some components along the other vectors  $A^\gamma b$  for  $\gamma \neq \alpha$ . This is a maximization of a component of the vector  $\xi$ , but it is a component as seen

in a nonorthogonal set of coordinates. Hence, the upper bounds obtained in the various directions define a parallelogram (more generally an  $n$  dimensional paralleliped) which can be considered as an approximation to the recovery region, as shown in Fig. 2. As before there is some point on each side of the parallelogram which is in the recovery region, but no point outside the parallelogram is in the region.

The minimum distance to a side of the parallelogram, i.e. the minimum perpendicular distance to a side, is an approximation  $\rho^*$  to the degree of controllability,  $\rho$ . When the system becomes uncontrollable, the columns of  $Q$  become linearly dependent, and hence the perpendicular distance to one of the sides becomes zero. This means that this  $\rho^*$  has the essential property that  $\rho^*=0$  whenever  $\rho=0$ .

We conclude that for the scalar control case we have a viable method of approximating the degree of controllability. A simple method will be presented in a later section to determine the needed minimum perpendicular distance.

This approximation is still an upper bound, and it can be improved, in fact made arbitrarily good, by considering more directions in the state space. Let  $e$  be any desired unit vector expressed as a column matrix of components. By examining (12) the state in the recovery region having a maximum component along the direction  $e$  is obtained using the control

$$u = -\text{sgn}[e^T Q \psi] \quad (14)$$

and hence no points in the recovery region lie beyond the line perpendicular to  $e$  and a distance

$$\int_0^T |e^T Q \psi| dt \quad (15)$$

from the origin (but at least one point in the recovery region lies on the line). Figure 3 illustrates how use of three  $e$ 's ( $e_I$ ,  $e_{II}$ , and  $e_{III}$ ) identifies three tangents to the recovery region, and when taken together they begin to approximate the region boundary. Let  $\beta$  be the minimum value of (15) for any set of directions  $e$  considered. Then an improved estimate of the degree of controllability is  $\rho^{**} = \min(\rho^*, \beta)$ , and  $\rho^{**} > \rho$  can be made arbitrarily close to the true degree of controllability  $\rho$  by picking a sufficient number of directions  $e$ . This method of improving the approximation to the degree of controllability will also be generalized to the multiple control case.

#### FUNDAMENTAL EQUATION FOR RECOVERY REGION APPROXIMATION IN THE MULTIPLE CONTROL CASE

The previous section presented a procedure for generating an approximation  $\rho^{**}$  to the degree of controllability  $\rho$  in the case of a scalar control  $u$ . The procedure required the use of  $n$  carefully chosen directions in the state space,  $b, Ab, \dots, A^{n-1}b$ , in the approximation to the recovery region in order to insure that  $\rho^{**}$  had the property that  $\rho^{**}=0$  if and only if the system is uncontrollable. If the control vector is  $m$  dimensional with  $m>1$  it is no longer obvious how to obtain this property, since the columns of the  $Q$  matrix necessarily contain linearly dependent vectors. Instead, we will consider the eigenvectors and generalized eigenvectors of the  $A$  matrix of (2) as the  $n$  linearly independent directions in the state space. Certainly some modifications must apply when these vectors are complex. In the single control case the value of  $\rho^{**}$  became zero when the system became uncontrollable because linear dependence of the vectors  $b, Ab, \dots, A^{n-1}b$  implies the collapse of at least one

dimension of the paralleliped. The vectors chosen here for the multidimensional control case do not exhibit this reduction. Nevertheless, it will be shown in the next section that the desired property of the resulting  $p^{**}$  can be demonstrated under fairly general assumptions. This section is devoted to generating the appropriate expression for  $\xi$ , equivalent to equations (4,10-12), expressed in terms of components in these eigenvector directions.

Let  $J$  be the Jordan canonical form of the matrix  $A$ , and let  $P$  be the matrix of eigenvectors and generalized eigenvectors so that

$$P^{-1}AP = J \quad (16)$$

$$J = \text{diag}(J_1, J_2, \dots, J_r)$$

where the  $J_i$  are the square Jordan blocks of dimension  $\mu_i$ . Associated with each block is an eigenvalue  $\lambda_i$ ,  $i=1,2,3,\dots,r$ , so that  $r \leq n$  and  $r$  is greater than or equal to the number of distinct eigenvalues. Also, let  $p_j$  be the  $n$  columns of  $P$ , and  $q_j^T$  be the  $n$  rows of  $P^{-1}$  (the left eigenvectors and generalized left eigenvectors).

The desired fundamental equation for  $\xi$  is given in the following theorem. The theorem is made significantly more complicated in order to handle repeated roots, but it is necessary to treat such roots since the rigid body modes for any spacecraft involve double roots.

**Theorem 1:** The displacement  $\xi = x(0) - x(T)$  after time  $T$  of the system state for equation (2) resulting from control  $u(t)$ ,  $t \in [0, T]$ , can be written as

$$-\xi = \int_0^T \sum_{j=1}^n [h_j(l_j^T u)] p_j dt \quad (17)$$

where  $l_j$  is a possibly time dependent vector

$$l_j(t) = [l_{j1} \ l_{j2} \ \dots \ l_{jm}]^T \quad (18)$$

$$l_{j\beta}(t) = a_{j\beta} + l_{j+1,\beta}(-t) + \frac{1}{2!} a_{j+2,\beta}(-t)^2 + \dots + \frac{1}{(k_i-j)!} a_{k_i,\beta}(-t)^{k_i-j} \quad (19)$$

$$a_{j\beta} = q_j^T b_\beta \quad (20)$$

with  $b_\beta$  given in (6). The values of  $k_i$  are determined by the dimensions of the Jordan blocks as follows

$$k_0 = 0 \quad ; \quad k_i = \sum_{l=1}^i \mu_l \quad (21)$$

and the values of  $j$  associated with each  $i=1,2,\dots,r$  are

$$j = k_{i-1} + 1, k_{i-1} + 2, \dots, k_i \quad (22)$$

The  $h_j$  are given by

$$h_j = e^{-\lambda_i t} \quad (23)$$

for  $i$  and  $j$  related by (22).

For the special case of all distinct eigenvalues ( $r=n$ ) equation (17) simplifies significantly, since there is no need for a distinction between  $i$  and  $j$ , and the components of  $l_j$  become the constants  $l_{j\beta} = a_{j\beta} = q_j^T b_\beta$ . Then (17) becomes

$$-\xi = \int_0^T \sum_{j=1}^n e^{-\lambda_j t} (q_j^T B u) p_j dt \quad (24)$$

Due to space limitations the proof of this theorem is omitted, but can be found in reference [2] or the journal version of this paper (to appear).

## FUNDAMENTAL EQUATION IN TERMS OF REAL VALUED FUNCTIONS

Take one eigenvalue from each complex conjugate pair and assign the  $j$  or  $j'$ 's associated with its eigenvector or generalized eigenvectors,  $p_j$ , to the set  $C$ . Let  $C^*$  represent the set of  $j'$ 's associated with real eigenvalues.

For any complex eigenvalue the associated  $p_j$ ,  $h_j$ , and  $l_j$  in (17) will be complex valued. Let their real and imaginary parts be given as follows together with the definitions of  $\phi_j$  and  $\eta_j$ :

$$\begin{aligned} p_j &= r_j + is_j \\ h_j &= e_j + ig_j \\ l_j &= \gamma_j + i\delta_j \quad j=1, 2, \dots, n \\ \phi_j^T &= e_j \gamma_j^T - g_j \delta_j^T \\ \eta_j &= e_j \delta_j^T + g_j \gamma_j^T \end{aligned} \quad (25)$$

The vectors representing the semiaxes of the parallelopiped are given by

$$\left( \int_0^T \sum_{\beta=1}^n 2|\phi_{j\beta}| dt \right) r_j \quad j \in C \quad (26)$$

$$\left( \int_0^T \sum_{\beta=1}^n 2|\eta_{j\beta}| dt \right) s_j \quad j \in C \quad (27)$$

$$\left( \int_0^T \sum_{\beta=1}^n |\phi_{j\beta}| dt \right) r_j \quad j \in C^* \quad (28)$$

(Note that these results can now be applied without regard for the reordering of the eigenvectors used in their derivation.)

Let this set of  $n$  vectors be denoted by  $v_1, v_2, \dots, v_n$ . The minimum perpendicular distance to a side of this parallelopiped can be used as an approximation to the degree of controllability, and as before the approximation can be made arbitrarily tight by using distances in additional directions in the state space.

Define the matrix  $F = [v_1 \ v_2 \ \dots \ v_n]$ . Each  $v_\alpha$  vector goes from the origin, or center of the parallelopiped volume, to the center of one of its sides. Any surface of the parallelopiped consists of edges that are parallel to  $n-1$  of the  $v$  vectors. Let  $d_j$ , corresponding to  $v_j$ , represent the perpendicular distance from the origin to that surface for which no edge is parallel to  $v_j$ . Then  $d_j$  is the component of  $v_j$  normal to this surface.

**Theorem 2:** The normal distances  $d_j$ ,  $j=1, 2, \dots, n$ , to the surfaces of the  $n$  dimensional parallelopiped prescribed by  $F$  are the reciprocals of the magnitudes of the column vectors given by  $(F^T)^{-1}$ .

Proof omitted due to space limitations (see [2] or journal version).

The approximation  $\rho^*$  of the degree of controllability  $\rho$  from the parallelopiped bound on the recovery region is

$$\rho^* = \min_j d_j \quad (29)$$

This approximate degree of controllability can be made arbitrarily tight by including the recovery region bounds in other directions as well as those of parallelopiped axes. This approximation  $\rho^*$  will go to zero when the system becomes uncontrollable as indicated in the following theorem.

**Theorem 3:** Suppose that all eigenvalues of the matrix  $A$  for system (2)

are distinct. Then  $\rho^*=0$  if and only if  $\rho=0$ . That is, the approximate degree of controllability based on the minimum perpendicular distance to the surface of the parallelogram  $F$  obtained from (26-28) will be zero if and only if the system is uncontrollable.

Proof omitted due to space limitations (see [2] or journal version).

#### SPECIALIZATION OF THE DEGREE OF CONTROLLABILITY CRITERION TO MODAL COORDINATES

Consider a lightly damped flexible spacecraft with dynamic equations expressed in terms of spacecraft normal modes. Temporarily consider the shape control problem alone so that the rigid body mode is neglected. Then the equations are

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \Gamma_i^T u \quad i=1,2,\dots,(n/2) \quad (30)$$

The  $\Gamma_i$  and  $u$  are defined so that each component of  $u$  is bounded by unity. Let the numbers  $N_i$  represent the relative importance we assign to the control of the  $\eta_i$ 's. We must also specify the importance we assign to the control of  $\dot{\eta}_i$  in the state space which will be generated. Since the system is lightly damped, if  $\eta_i$  is  $\sin(\omega_i t + \phi)$ , then  $\dot{\eta}_i$  will be  $\omega_i \cos(\omega_i t + \phi)$  approximately, so that the relative importance of controlling  $\dot{\eta}_i$  is represented by  $\omega_i N_i$ . Now generate the state variables as follows

$$x_{2i-1} \triangleq \eta_i/N_i \quad ; \quad x_{2i} \triangleq \dot{\eta}_i/\omega_i$$

so that all unit deviations from the origin of the state space will be considered equally serious. Then

$$\begin{bmatrix} \dot{x}_{2i-1} \\ \dot{x}_{2i} \end{bmatrix} = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\zeta_i\omega_i \end{bmatrix} \begin{bmatrix} x_{2i-1} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma_i^T/(N_i\omega_i) \end{bmatrix} u \quad (31)$$

Let the coefficient matrices be  $A_i$  and  $B_i$ . Then the normalized system equation (2) has coefficient matrices  $A$  and  $B$ , and eigenvector matrix  $P$ , which can be partitioned as follows

$$\begin{aligned} A &= \text{diag} [A_1, A_2, \dots, A_{n/2}] \\ B &= [B_1^T \ B_2^T \ \dots \ B_{n/2}^T]^T \\ P &= \text{diag} [P_1, P_2, \dots, P_{n/2}] \\ P^{-1} &= \text{diag} [P_1^{-1}, P_2^{-1}, \dots, P_{n/2}^{-1}] \end{aligned} \quad (32)$$

where

$$P_i = \begin{bmatrix} 1 & 1 \\ \lambda_{2i-1}/\omega_i & \lambda_{2i}/\omega_i \end{bmatrix} \quad ; \quad P_i^{-1} = \frac{1}{\lambda_{2i} - \lambda_{2i-1}} \begin{bmatrix} \lambda_{2i} & -\omega_i \\ -\lambda_{2i-1} & \omega_i \end{bmatrix}$$

$$\lambda_{2i-1} = \omega_i(-\zeta_i + i\sqrt{1-\zeta_i^2}) \quad ; \quad \lambda_{2i} = \omega_i(-\zeta_i - i\sqrt{1-\zeta_i^2})$$



Using the rows of  $P^{-1}$  and the columns of  $B$  in  $l_{j\beta} = a_{j\beta} = q_j^T b_\beta$ , gives a zero real part ( $\gamma_{2i,\beta} = 0$ ), and imaginary part

$$\delta_{2i,\beta} = \Gamma_{i\beta} / (2N_i \omega_i \sqrt{1-\zeta_i^2})$$

where  $\Gamma_{i\beta}$  is the  $\beta^{\text{th}}$  component of  $\Gamma_i$ . We wish to calculate (26) and (27) for  $j = 0$ , i.e. for one of the complex roots in each complex conjugate pair. From (25)  $\phi_{2i,\beta} = -g_{2i} \delta_{2i,\beta}$ , and  $g_{2i}$  is the imaginary part of  $\exp[-\lambda_{2i} t]$ .

Then (26) becomes

$$\left( \int_0^T \sum_{\beta=1}^m \frac{|\Gamma_{i\beta}|}{N_i \omega_i \sqrt{1-\zeta_i^2}} e^{\zeta_i \omega_i t} |\sin(\sqrt{1-\zeta_i^2} \omega_i t)| dt \right) r_{2i} \quad (33)$$

For a lightly damped system the exponential will not change significantly during one oscillation of the sine wave. Assuming that  $T$  is long compared with the period of the sine wave (so that the effect of partial completion of the final period of oscillation of the sine wave is negligible), the absolute value of the sine can be replaced by  $2/\pi$ , its average over a period. Then (33) becomes

$$\frac{2}{\pi} \frac{\|\Gamma_i\|_A}{N_i \omega_i} \left( \frac{e^{\zeta_i \omega_i T} - 1}{\zeta_i \omega_i \sqrt{1-\zeta_i^2}} \right) r_{2i} \quad (34)$$

$$\|\Gamma_i\|_A \triangleq \sum_{\beta=1}^m |\Gamma_{i\beta}| \quad (35)$$

The analogous calculation for (27) gives the same result with  $r_{2i}$  replaced by  $s_{2i}$ . Together these vectors form the set  $v_1, v_2, \dots, v_n$  which are the columns of  $F$ .

The  $r_{2i}$  and  $s_{2i}$  are the real and imaginary parts of the appropriate column of  $P$ . Looking only at the appropriate partition of  $F$ , call it  $F_i$ , we have

$$F_i = \sigma \begin{bmatrix} 1 & 0 \\ -\zeta_i & -\sqrt{1-\zeta_i^2} \end{bmatrix} ; F_i^{-1} = \frac{1}{\sigma \sqrt{1-\zeta_i^2}} \begin{bmatrix} \sqrt{1-\zeta_i^2} & 0 \\ -\zeta_i & -1 \end{bmatrix}$$

where  $\sigma$  is the coefficient of  $r_{2i}$  in (34). The associated values of  $d_j$  from Theorem 2 result from taking the magnitude of the rows of  $F_i^{-1}$  and are given by

$$d_{2i-1} = \sigma ; d_{2i} = \sigma \sqrt{\frac{1-\zeta_i^2}{1+\zeta_i^2}}$$

the second of which is the smaller. From (29) the approximate degree of controllability  $\rho^*$  is

$$\rho^* = \min_i \left[ \frac{2}{\pi} \frac{\|\Gamma_i\|_A}{N_i \omega_i} \left( \frac{e^{\zeta_i \omega_i T} - 1}{\zeta_i \omega_i \sqrt{1+\zeta_i^2}} \right) \right] \quad (36)$$

Note that when damping is present, the recovery region expands with  $T$  in such a way that  $\rho^*$  grows exponentially. Furthermore, when the natural frequency of the minimizing mode is decreased the approximate degree of controllability increases.

An important special case is that of a system model which has no

damping. The approximate degree of controllability in this case becomes

$$\rho^* = \frac{2T}{\pi} \min_i \frac{\|\Gamma_i\|_A}{N_i \omega_i} \quad (37)$$

Note that  $\rho^*$  increases linearly with time, making  $T$  a scale factor which can be dropped. We conclude that the approximate degree of controllability is simply the minimum norm (in the sense of (35)) of the rows of the suitably normalized control coefficient matrix when the system is represented in modal form. The normalization in (37) is that applying to the derivative of the modal coordinate  $\eta_i$ .

Now let us introduce the rigid body mode into the problem so that the control objective is simultaneous attitude and shape control of the flexible vehicle. For simplicity consider only one such mode, and let the variable involved be  $\theta$ . Let the normalization representing our degree of interest in controlling  $\theta$  be  $N_\theta$ , and that for  $\dot{\theta}$  be  $N_{\dot{\theta}}$ . Then define state variables  $x_\theta = \theta/N_\theta$  and  $x_{\dot{\theta}} = \dot{\theta}/N_{\dot{\theta}}$ , and the equation  $\ddot{\theta} = \Gamma_\theta u$  becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & N_{\dot{\theta}}/N_\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma_\theta^T/N_{\dot{\theta}} \end{bmatrix} u$$

and

$$P = \begin{bmatrix} 1 & 0 \\ 0 & N_{\dot{\theta}}/N_\theta \end{bmatrix}; \quad P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & N_\theta/N_{\dot{\theta}} \end{bmatrix}$$

From (20) and (19),  $\alpha_{\theta\theta} = 0$ ,  $\alpha_{\dot{\theta}\theta} = \Gamma_{\theta\theta}/N_\theta$ ,  $\lambda_{\theta\theta} = -(\Gamma_{\theta\theta}/N_\theta)t$ ,  $\lambda_{\dot{\theta}\theta} = \Gamma_{\theta\theta}$ , and  $h_\theta = h_{\dot{\theta}} = 1$ . Then (28) gives

$$\frac{\|\Gamma_\theta\|_A}{N_\theta} \frac{T^2}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \frac{\|\Gamma_\theta\|_A}{N_{\dot{\theta}}} T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (38)$$

Calculating  $d_\theta$  and  $d_{\dot{\theta}}$  using Theorem 2 results in the coefficients of these two vectors.

The approximate degree of controllability  $\rho^*$  when the rigid body mode is included is the minimum of the  $\rho^*$  given by (37) (or (36) when damping is included) and the two coefficients of the vectors in (38):

$$\rho^* = \min \left[ \left( \frac{2T}{\pi} \right) \min_i \frac{\|\Gamma_i\|_A}{N_i \omega_i}, \left( \frac{T^2}{2} \right) \frac{\|\Gamma_\theta\|_A}{N_\theta}, T \frac{\|\Gamma_\theta\|_A}{N_{\dot{\theta}}} \right] \quad (39)$$

Note that because of the modal representation used in the system equations, there is no coupling of the weighting factors  $N_i$ ,  $N_\theta$ , and  $N_{\dot{\theta}}$  from one mode to another. Thus, if one is not particularly interested in controlling any specific mode, the corresponding  $N$  can be made very small. Then that particular mode is simply neglected in the minimization to determine  $\rho^*$ . For example, in some applications one might feel that it is important to control the rigid body angle  $\theta$ , but that if  $\theta$  is controlled well the value of  $\dot{\theta}$  is unimportant. In such a case one simply ignores the  $T \|\Gamma_\theta\|_A / N_{\dot{\theta}}$  term in (39).

## EXAMPLES

The methods of this paper were applied to an example flexible spacecraft. Due to space limitations only a few results will be cited. The spacecraft consists of a rigid symmetric central hub with two 60 foot radial stem type booms. The two lowest appendage modes are used to generate two symmetric and two antisymmetric spacecraft modes. Both torque and force actuators were considered. Figure 4 applies to antisymmetric spacecraft modes and plots the values of  $\| \Gamma_i \|_A / \omega_i$ , normalized by the torque actuator strength. Equation (37) shows that the degree of controllability is calculated by scaling each curve of Fig. 4 according to the normalization  $N_i$  and the actuator strength, and taking the minimum of these curves at the actuator location. The optimum location for the torque actuator is easily determined as that point which maximizes this minimum. Any regions where one of the curves is near zero should be avoided. For a system which truly has only two modes, and with unit normalizations, the best torquer location is at the end of the appendage.

## REFERENCES

1. R.W. Longman and K.T. Alfriend, "Actuator Placement from Degree of Controllability Criteria for Regular Slewing of Flexible Spacecraft," IAF paper No. 79-195, 30th Congress of the International Astronautical Federation, Munich, Germany, September 16-22, 1979.
2. C.N. Viswanathan, Aspects of Control of Large Flexible Spacecraft, Ph.D. Dissertation, Department of Mechanical Engineering, Columbia University, 1979.

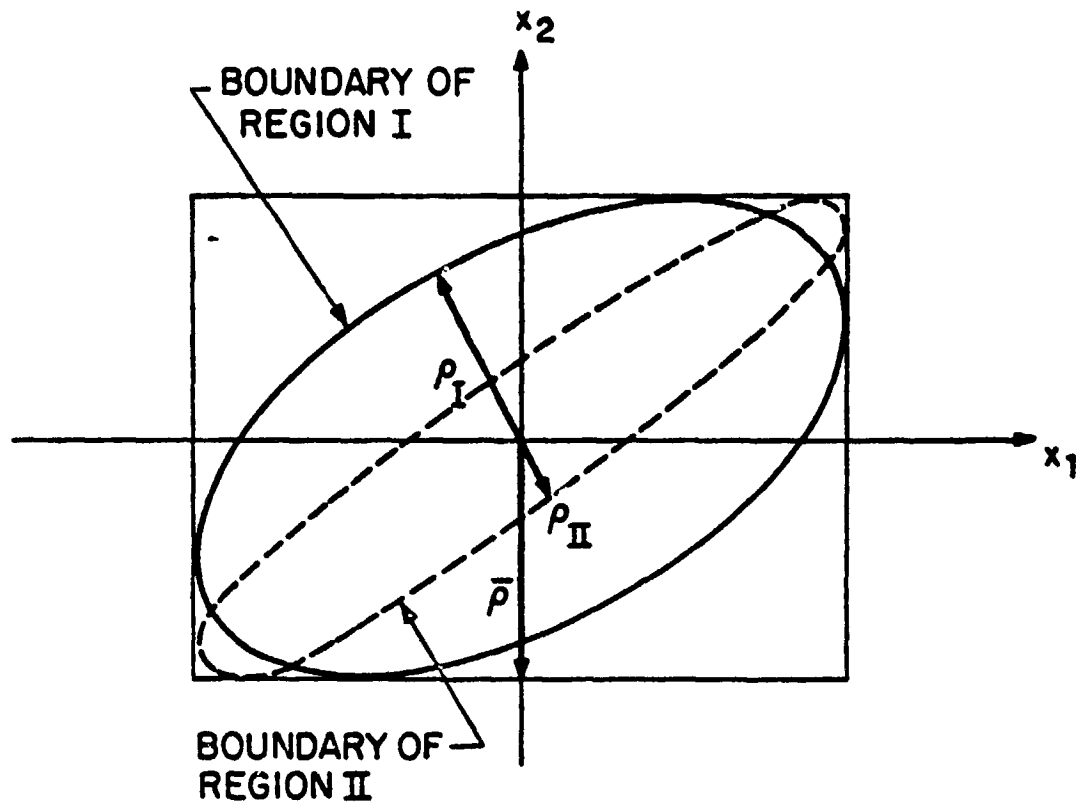


Fig. 1. The recovery region and its rectangular approximation

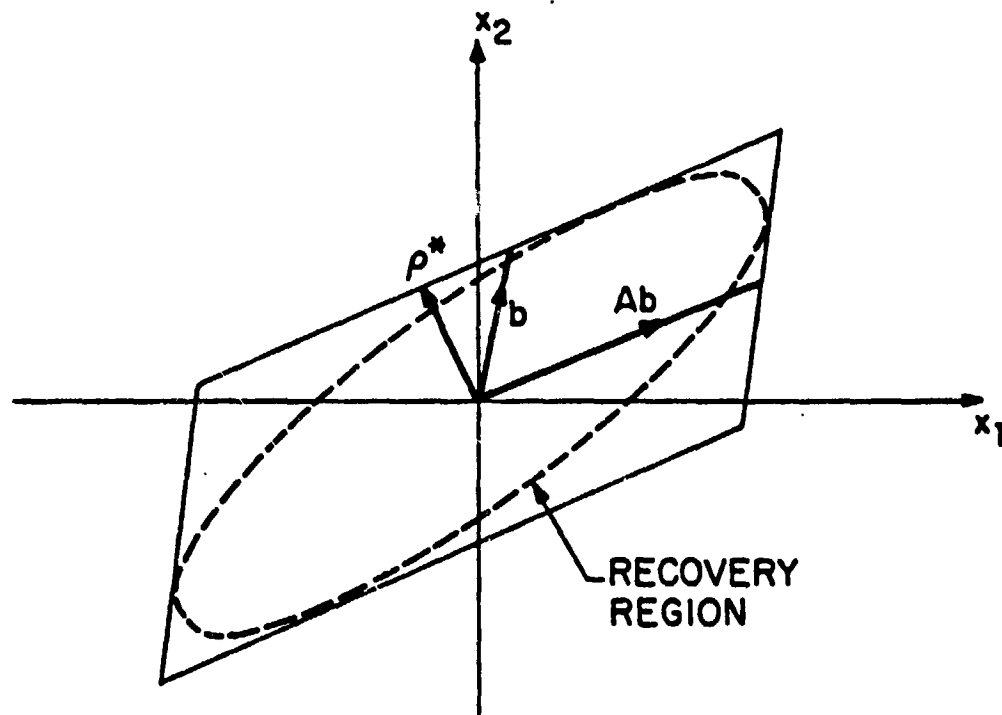


Fig. 2. Parallelogram bound on the recovery region

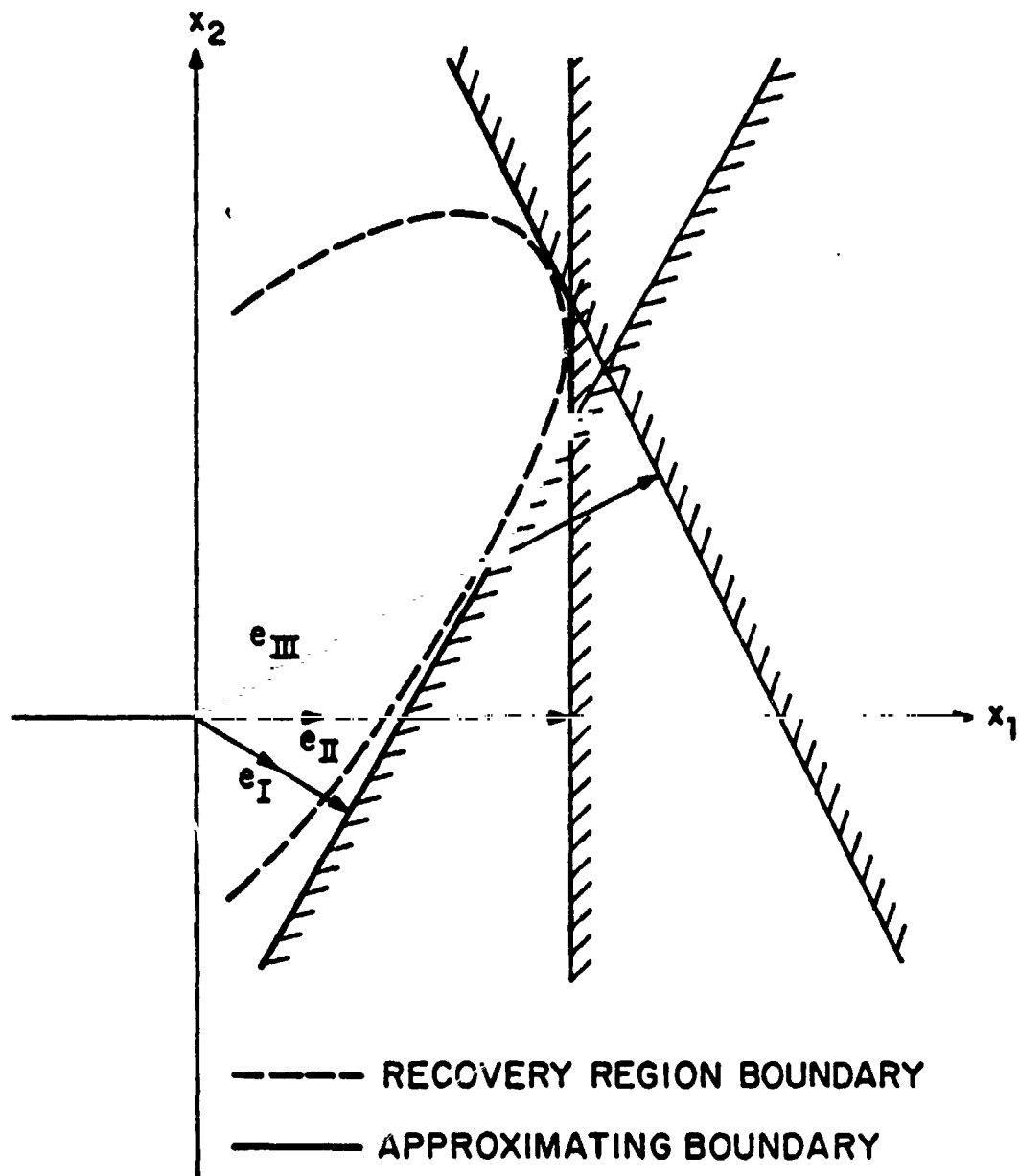


Fig. 3. Improving the approximation to the recovery region

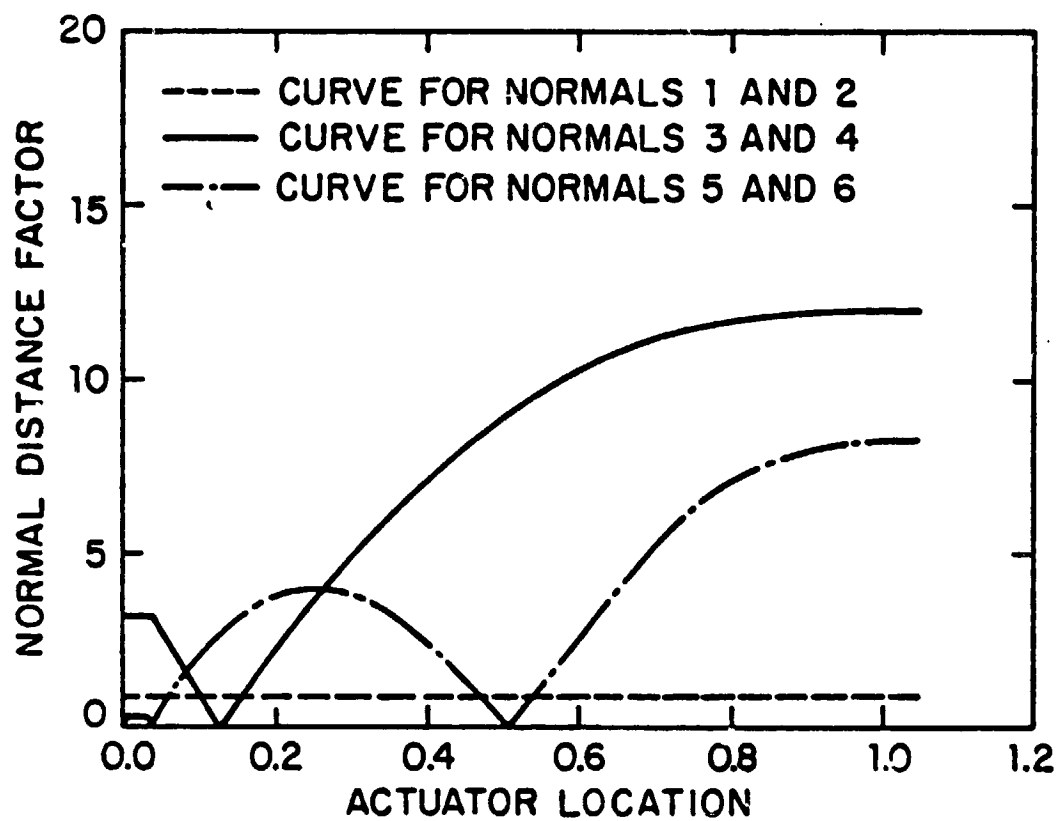


Fig. 4. Determination of the degree of controllability as a function of actuator location

## PART II

INSTRUMENT FLEXIBILITY LIMITATIONS FOR  
STABILITY OF MULTI-PURPOSE SPACECRAFT  
WITH FIXED CONTROL SYSTEM DESIGN

## 9. ANALYSIS

### 9.1 Introduction

In part Two, attention is focused on a component of a spacecraft while keeping the control system design fixed. These spacecraft are so designed that certain of their components are interchangeable with other physically compatible components. These interchangeable components may be designed to accomplish different tasks. These spacecraft can, therefore, be called multi-purpose spacecraft.

A typical example is a Shuttle-based Instrument Pointing System (IPS). Such a spacecraft will have a Space-Shuttle with a multi-purpose device called the Instrument Pointing Mount (IPM) to which one of a family of scientific instruments is attached. The instruments are typically lightweight and their flexibility becomes significant due to pointing requirements. The shuttle and IPM are modeled as rigid components of the idealized system, while the instrument is modeled as a flexible component. A control system can be specifically designed for such an IPS with a given instrument. However, the central idea of the IPM is to make it a multi-purpose device, so that it works not simply for one instrument but for a whole series of instruments, many of which have yet to be designed. In fact, once the IPM characteristics are defined, the scientific instruments have to be designed so that they will function properly when attached to the IPM. This is the inverse of the usual design problem, in which the physical plant is given and the



control system is to be determined. Instead, here, the physical plant is variable because of the interchangeable instrument. The approach taken, therefore, is to predesign a comprehensive control system which is shuttle-based, and hence remains unchanged for the spacecraft for a whole series of instruments that might be attached to the IPM. Severe accuracy requirements in the orientation of these shuttle-based instruments, sometimes to within fractions of an arcsecond, may be demanded of the control system design. And usually a dual control system will need to be adopted with one control system located in the shuttle, and the other in the IPM (for fine control). In this case, the IPM can have motion relative to the shuttle. Now, the question is "what kind of instruments are stably controllable with any given control system?" In response to this, it is best to describe these instruments in the most general terms possible so as to allow the designer ample freedom in their design.

## 9.2 Parameter Plane Technique

This method has its origins about a century ago in the idea of investigating the system response characteristics [8] by algebraic approach. In a wide variety of control problems the designer is interested not only in the stability of the system but also in the essential features of the system behavior over time. In its original form the approach began by treating two coefficients of a characteristic equation as variables, and studying how the roots of the equation are affected when these two coefficients are changed. By plotting the characteristic curves in the plane of the variable coefficients, the method enables adjustment of these coefficients so that the roots of the characteristic equation may be set at any desired locations. After the curves are plotted, the variable coefficients can be adjusted without any calculations. Several researchers have since then extended the method and its applications to various problems such as sensitivity analysis of linear control systems, circuit synthesis, steady-state response analysis, sampled-data linear systems, and to other related problems of linear system design. Also, the method has been successfully applied to non-linear systems. The parameter plane technique for analysis and synthesis of linear and nonlinear control systems is amply described in Siljak's monograph [9]. Once the system characteristic equation has been obtained, the parameter plane method enables

the designer to evaluate graphically the roots of the equation. Hence, he may design the control system in terms of the chosen performance criteria; e.g., absolute stability, damping ratio, and settling time. Thus, the technique is the mapping of the roots of the characteristic equation from the complex plane onto what is known as a parameter plane. The design procedure has also been simplified by Siljak who introduced Chebyshev functions into the equations, thereby putting them in a suitable form for digital computer simulation.

The parameter plane method requires that the control system be described by a characteristic equation which is transformed into the complex domain (s-domain). Two adjustable parameters,  $\alpha$ ,  $\beta$ , (which are of interest to the designer) are selected, and the characteristic equation (essentially its coefficients) is recast in terms of  $\alpha$ ,  $\beta$ . Suppose the characteristic equation (CE) is

$$CE = \sum_{j=0}^n f_j s^j = 0 \quad (9.2-1)$$

then

$$f_j = f_j(\alpha, \beta), \quad j = 0, 1, 2, \dots, n \quad (9.2-2)$$

The points in the s-plane (complex domain) are described best suited to this method by the two system characteristic quantities  $\zeta$ ,  $\omega_n$ , where  $\zeta$  is the damping ratio ( $0 \leq \zeta \leq 1$ ) and  $\omega_n$  is the undamped natural frequency. Let

$$s = x + i y = -\zeta \omega_n + i \omega_n \sqrt{1-\zeta^2} \quad (9.2-3)$$

Any power of s can be written as

$$s^j = X_j + i Y_j \quad (9.2-4)$$

where  $X_j$ ,  $Y_j$  are the real and imaginary parts for which there exist the following relationships

$$Z_j = -2\zeta\omega_n Z_{j-1} - \omega_n^2 Z_{j-2}$$

$$j = 2, 3, 4, \dots$$

$$Z_j = X_j \text{ or } Y_j$$

$$X_0 = 1; \quad X_1 = -\zeta\omega_n$$

$$Y_0 = 0; \quad Y_1 = \omega_n \sqrt{1-\zeta^2} \quad (9.2-5)$$

Substituting (9.2-4) in (9.2-1)

$$CE = \sum_{j=0}^n f_j (X_j + i Y_j) = 0 \quad (9.2-6)$$

from which we obtain

$$\sum_{j=0}^n f_j X_j = 0$$

$$\sum_{j=0}^n f_j Y_j = 0 \quad (9.2-7)$$

Now, the coefficients  $f_j$  may be linear or nonlinear in the parameters  $\alpha, \beta$  depending on the choice of the parameters. Consider a linear form for the  $f_j$ , i.e., let

$$f_j = a_j \alpha + b_j \beta + c_j \quad (9.2-8)$$

where  $a_j, b_j, c_j$  are constants. Then

$$\sum_{j=0}^n f_j Z_j = \sum_{j=0}^n (a_j \alpha + b_j \beta + c_j) Z_j$$

$$= \left( \sum_{j=0}^n a_j Z_j \right) \alpha + \left( \sum_{j=0}^n b_j Z_j \right) \beta + \left( \sum_{j=0}^n c_j Z_j \right)$$

$$= 0 \quad (9.2-9)$$

in which  $Z_j = X_j$  or  $Y_j$ . We define

$$A = \sum_{j=0}^n a_j x_j ; B = \sum_{j=0}^n b_j x_j ; C = \sum_{j=0}^n c_j x_j \quad (9.2-10)$$

and similarly,  $A'$ ,  $B'$ ,  $C'$  for  $Y_j$ , then from (9.2-9)

$$A\alpha + B\beta + C = 0$$

$$A'\alpha + B'\beta + C = 0 \quad (9.2-11)$$

The coefficients  $A$ ,  $B$ ,  $C$  etc. are determined if  $f$  and  $\omega_n$  are chosen. These two equations can then be solved for  $\alpha$  and  $\beta$  if their discriminant is nonzero.

In a similar fashion, two equations nonlinear in  $\alpha$  and  $\beta$  can be obtained if  $f_j$  are nonlinear functions in  $\alpha$ ,  $\beta$ .

In this case, there will be multiple images in the parameter plane corresponding to a root in the complex plane defined by  $f$ ,  $\omega_n$ .

The characteristic curves in the parameter plane are plotted for various values of  $f$  and  $\omega_n$ . These curves are called  $f$ -curves and  $\omega_n$ -curves, because a set of curves can be plotted, each curve corresponding to a fixed value of  $f$  (or  $\omega_n$ ) and the other quantity  $\omega_n$  (or  $f$ ) varying along the curve. Similar curves are plotted for real roots  $\sigma$  of the CE. For real roots one of the equations (connected with the imaginary part  $Y_j$ ) in (9.2-11) vanishes, and for various values of  $\sigma$  we obtain the  $\sigma$ -lines in the parameter plane. Having plotted these curves one can read the roots of the CE corresponding to any point in the  $\alpha$ - $\beta$  plane. Generally, these curves are shaded to help relate the crossing of these curves in the

parameter plane to crossing of the axes in the complex plane. The details on the shading convention adopted for the parameter plane curves can be found in Siljak's monograph [9].

One of the most important regions in the parameter plane is the unstable region which the designer must avoid in designing the system. But, though stability is necessary in a wide variety of control problems it is not sufficient, and the designer might be interested in other features of the system behavior. The parameter plane can be used in several ways as one wishes to study the system response. Thus, for instance, a designer might want to keep  $\zeta > 0.2$  for all the roots of the equation, and a similar restriction on  $\omega_n$ . Depending on his requirements he can mark out the regions of interest and adjust his parameters  $\alpha, \beta$  to obtain best performance for the system.

The following are the main steps in the application of the parameter plane technique:

- 1) Obtain the characteristic equation (CE) for the system in the s-domain, i.e.,  $\sum_{j=0}^n f_j s^j = 0$ .
- 2) Identify two parameters of interest  $\alpha, \beta$  and recast the coefficients  $f_j$  of the CE in terms of  $\alpha, \beta$ .
- 3) Use the substitution  $s^j = X_j + i Y_j$  and obtain two algebraic equations,  $\sum_{j=0}^n f_j X_j = 0$ , and  $\sum_{j=0}^n f_j Y_j = 0$ .

- 4) Using the recursive relationships from (9.2-5) for  $X_j$ ,  $Y_j$  in terms of  $\xi$ ,  $\omega_n$ , solve the two algebraic equations for  $\alpha$ ,  $\beta$  for various values of  $\xi$ ,  $\omega_n$ .
- 5) Plot the  $\xi$ -curves or  $\omega_n$ -curves in the parameter plane ( $\alpha$ - $\beta$  plane).
- 6) For real roots use the CE with  $s = \sigma$ , and for various values of  $\sigma$  plot the  $\sigma$ -lines in the  $\alpha$ - $\beta$  plane.
- 7) Having plotted these curves mark the unstable regions in the parameter plane, and use the shading convention.

In the next chapter we will study the flexibility characteristics of the shuttle-based instrument of a multi-purpose spacecraft applying the parameter plane analysis.

## 10. ANALYSIS OF A SHUTTLE-BASED INSTRUMENT POINTING SYSTEM OF A MULTI-PURPOSE SPACECRAFT

### 10.1 Introduction

The parameter plane analysis has been applied to study stability of attitude control of spinning skylab [10], [11], and recently Seltzer and Shelton [12] applied the analysis to study the rigidity of spacecraft flexible appendages for a two-body model, a rigid body and a flexible instrument, with a spacecraft attitude controller of a standard position-integral-derivative (PID) state feed back type.

In the present work we will analyze a three-body system, a rigid shuttle, a rigid instrument pointing mount (IPM) and a flexible instrument. There will be two PID controllers, one shuttle-based and the other IPM-based. Once the equations are derived two cases will be examined: 1) the IPM is locked to the shuttle and only the shuttle-based PID controller is in operation, thus reducing this system essentially to a two-body system, and 2) the IPM is allowed to have motion relative to the shuttle with its PID controller also in operation, which is a three-body system. The model of the two-body system here differs from the model in [12] in its feed back control law.



## 10.2 Planar Model

Figure 12 shows a typical planar model of a three-body system in which  $B_{R1}$  is the rigid shuttle,  $B_{R2}$  is the rigid IPM and  $B_F$  is the flexible instrument whose characteristics are of interest to us. The system is inertially at rest and the rotational dynamics is assumed to be confined to a single plane of motion with control torques applied in this plane. Two attitude controllers of the proportional-integral-differential (PID) type are assumed, one located in the shuttle ( $B_{R1}$ ) and the other in the IPM ( $B_{R2}$ ). The shuttle-based controller applies torque  $\tau_1$  to  $B_{R1}$  and is meant for large attitude control, and the IPM-based controller applies an interaction torque,  $\tau_2$  to  $B_{R2}$  and  $-\tau_2$  to  $B_{R1}$ , and is meant for finer control of the instrument. The inertial attitude of  $B_{R1}$  is denoted by  $\Theta_1$  and the relative rotation of  $B_{R2}$  with respect to  $B_{R1}$  is denoted by  $\Theta_2$ . The flexible appendage is assumed to be an elastic beam and is characterized by distributed coordinates, and the hybrid coordinate approach [5] is used. In the stability analysis the appendage is characterized by a single assumed mode.

### 10.3 Dynamics

The kinetic energy of the system is derived in a similar manner as in part I, but is more involved due to the relative motion of  $B_{R2}$ . It is given by

$$\begin{aligned} T_s = & \dot{\Theta}_1^2 \left[ \gamma_1 + f_1(\Theta_2) + g_1(\Theta_2, \eta) + h_1(\eta) \right] \\ & + \dot{\Theta}_1 \dot{\Theta}_2 \left[ 2\gamma_2 + f_1(\Theta_2) + g_1(\Theta_2, \eta) + 2h_1(\eta) \right] \\ & + \dot{\Theta}_2^2 \left[ \gamma_2 + h_1(\eta) \right] + \dot{\Theta}_1 \dot{\Delta}_1 \left[ \gamma_3 + f_2(\Theta_2) \right] \\ & + \dot{\Theta}_2 \dot{\Delta}_1 \gamma_3 - \dot{\Theta}_1 \dot{\Delta}_3 - \dot{\Theta}_2 \dot{\Delta}_3 - \frac{1}{2\omega\mathcal{M}} \dot{\Delta}_1^2 + \frac{1}{2} \Delta_4 \end{aligned}$$

(10.3-1)

where

$$\begin{aligned} \gamma_1 = & \frac{1}{2} \mathcal{M}' (\gamma^D)^2 + \frac{1}{2} (\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_F) + \frac{\mathcal{M}'}{\mathcal{M}} (\psi_1 \gamma_1^D + \psi_2 \gamma_2^D) \\ & + \frac{m_{R1}}{\mathcal{M}} \psi_3 + m_F \rho \left( \gamma^c + \frac{1}{2} \rho \right) \\ \gamma_2 = & \frac{1}{2} (\mathcal{I}_2 + \mathcal{I}_F) + m_F \rho \left( \gamma^c + \frac{1}{2} \rho \right) - \frac{1}{2\omega\mathcal{M}} \psi_4 \\ \gamma_3 = & \frac{1}{\mathcal{M}} \psi_2 - \rho \end{aligned}$$

$$f_1(\Theta_2) = \frac{m_{R1}}{\mathcal{M}} \left[ \psi_1 (x_5 - x_4) + \psi_2 (x_6 - x_3) \right]$$

$$g_1(\Theta_2, \eta) = \frac{m_{R1}}{\mathcal{M}} \Delta_1 (x_5 - x_4)$$

$$h_1(\eta) = \frac{1}{2} \Delta_2 - \frac{1}{2\mathcal{M}} \Delta_1^2 - \frac{\psi_1}{\mathcal{M}} \Delta_1$$

$$f_2(\Theta_2) = \frac{m_{R1}}{\mathcal{M}} (x_3 - x_6)$$

$$\Delta_1 = \mathcal{M}_\phi \eta \quad ; \quad \dot{\Delta}_1 = \mathcal{M}_\phi \dot{\eta}$$

$$\Delta_2 = \eta^T \Delta \eta \quad ; \quad \dot{\Delta}_2 = \mathcal{M}_{\tau\phi} \dot{\eta}$$

$$\Delta_4 = \dot{\eta}^T \Delta \dot{\eta}$$

(10.3-2)

in which

$$x_3 = \alpha_2 \cos \Theta_2 - \alpha_1 \sin \Theta_2$$

$$x_4 = \alpha_1 \cos \Theta_2 + \alpha_2 \sin \Theta_2$$

$$x_5 = \gamma_1^D \cos \Theta_2 + \gamma_2^D \sin \Theta_2$$

$$x_6 = \gamma_2^D \cos \Theta_2 - \gamma_1^D \sin \Theta_2$$

(10.3-3)

$m_{R1}$ ,  $m_F$ ,  $\mathcal{M}$  are masses of  $B_{R1}$ ,  $B_F$  and total system, respectively,  $\mathcal{M}_\phi$ ,  $\mathcal{M}_{\tau\phi}$ ,  $\Delta$  are as given in (3.3-20) in part I,

and other quantities are related to the geometry and mass distribution in the system.

The potential energy of the system is only due to the elastic strain energy and is, as in part I, given by

$$V_s = \frac{1}{2} \eta^T \nu \eta \quad (10.3-4)$$

From the Lagrangian  $L = T_s - V_s$ , the equations of motion are derived, and after linearizing these equations about a nominal state  $\bar{\theta}_1, \bar{\theta}_2$ , and assuming  $\bar{\theta}_2 = \dot{\bar{\theta}}_2 = \ddot{\bar{\theta}}_2 = \bar{\theta}_1 = \dot{\bar{\theta}}_1 = \ddot{\bar{\theta}}_1 = 0$ , we obtain

$$\begin{bmatrix} 2p_3 & p_6 & p_8 \\ p_6 & 2\gamma_2 & (\mu_0\gamma_3 - \mu_{r0}) \\ p_8 & (\mu_0\gamma_3 - \mu_{r0}) & \mu' \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\eta} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \eta \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ 0 \end{bmatrix}$$

(10.3-5)

where only a single mode for the appendage has been used, and  $\theta_1, \theta_2$  are small perturbations about  $\bar{\theta}_1, \bar{\theta}_2$ , respectively,  $\lambda$  is a damping coefficient for the appendage,

$$\mu' = \Delta - \frac{\mu_\phi^2}{\mathcal{M}}$$

(10.3-6)

and  $p_3$ ,  $p_6$ ,  $p_8$  are system constants based on mass distribution and geometry. Now, let the torques  $\tau_1, \tau_2$  of the PID controllers be

$$\tau_1 = - \left[ k_{11} \theta_1 + k_{12} \dot{\theta}_1 + k_{13} \int_0^t \theta_1 dt \right]$$

$$\tau_2 = - \left[ k_{21} \theta_2 + k_{22} \dot{\theta}_2 + k_{23} \int_0^t \theta_2 dt \right]$$

(10.3-7)

Taking the transform of (10.3-5) and (10.3-7) into the s-domain (complex domain), we obtain

$$\begin{bmatrix} 2p_3 s^2 + k_{11} + k_{12}s + \frac{k_{13}}{s} & p_6 s^2 & p_8 s^2 \\ p_6 s^2 & 2p_2 s^2 + k_{21} + k_{22}s + \frac{k_{23}}{s} & (\mu_\phi \gamma_3 - \mu_{\gamma\phi}) s^2 \\ p_8 s^2 & (\mu_\phi \gamma_3 - \mu_{\gamma\phi}) s^2 & \mu' s^2 + \lambda s + \gamma \end{bmatrix}$$

$$\begin{bmatrix} \bar{\theta}_1(s) \\ \bar{\theta}_2(s) \\ \bar{\eta}(s) \end{bmatrix} = [0]$$

(10.3-8)

The characteristic equation for the system is obtained by equating the determinant of the coefficient matrix in this equation to zero. The result is

$$\sum_{j=0}^8 f_j s^j = 0 \quad (10.2-9)$$

where

$$f_0 = c_1 \nu$$

$$f_1 = c_2 \nu + c_1 \lambda$$

$$f_2 = c_3 \nu + c_2 \lambda + c_1 \mu'$$

$$f_3 = c_4 \nu + c_3 \lambda + c_2 \mu'$$

$$f_4 = c_5 \nu + c_4 \lambda + c_3 \mu'$$

$$f_5 = c_6 \nu + c_5 \lambda + c_4 \mu' - k_{13} \delta^2 \mu' - k_{23} p_s^2$$

$$f_6 = c_7 \nu + c_6 \lambda + c_5 \mu' - k_{11} \delta^2 \mu' - k_{21} p_s^2$$

$$f_7 = c_7 \lambda + c_6 \mu' - k_{12} \delta^2 \mu' - k_{22} p_s^2$$

$$f_8 = c_7 \mu' - 2 p_3 \delta^2 \mu'^2 - 2 p_3 p_6 \delta \mu' - 2 \gamma_2 p_8^2$$

(10.3-10)

in which

$$C_1 = K_{13} K_{23}$$

$$C_2 = K_{11} K_{23} + K_{13} K_{21}$$

$$C_3 = K_{11} K_{21} + K_{12} K_{23} + K_{13} K_{22}$$

$$C_4 = 2p_3 K_{23} + 2\gamma_2 K_{13} + K_{11} K_{22} + K_{12} K_{21}$$

$$C_5 = 2p_3 K_{21} + 2\gamma_2 K_{11} + K_{12} K_{22}$$

$$C_6 = 2p_3 K_{22} + 2\gamma_2 K_{12}$$

$$C_7 = 4\gamma_2 p_3 - p_6^2$$

$$\delta = \frac{\mu_{\tau p} - \mu_{\phi} \gamma_3}{\mu'} \quad (10.3-11)$$

The above characteristic equation is for a three-body system. The CE for a two-body system in which the IPM gimbal is locked to the shuttle so that  $\theta_2 = 0$ , and the PID controller in the IPM is absent (control gains  $K_{ij}$ 's for  $\tau_2$  are zero), is obtained by striking out the second row and column in the coefficient matrix in (10.3-8) as follows:

$$\begin{bmatrix} 2p_3 s^2 + K_{11} + K_{12} s + \frac{K_{13}}{s} & p_8 s^2 \\ p_8 s^2 & \mu' s^2 + \lambda s + \nu \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(s) \\ \bar{\eta}(s) \end{bmatrix} = [0] \quad (10.3-12)$$

from which the CE for a two-body system is

$$\text{where } \sum_{j=0}^5 f_j s^j = 0 \quad (10.3-13)$$

$$f_0 = a_I v$$

$$f_1 = a_P v + a_P \lambda$$

$$f_2 = a_D v + a_P \lambda + a_I \mu'$$

$$f_3 = v + a_D \lambda + a_P \mu'$$

$$f_4 = \lambda + a_D \mu'$$

$$f_5 = \mu' - \frac{b_s^2}{2b_3} \quad (10.3-14)$$

$$a_P = \frac{k_{11}}{2b_3}$$

$$a_D = \frac{k_{12}}{2b_3}$$

$$a_I = \frac{k_{10}}{2b_3}$$

(10.3-15)

At this point since we have obtained the CE in the s-domain we can apply the parameter plane analysis. We will consider a two-body system and a three-body system separately.



## 10.4 Two-Body System

Once the CE is obtained we have to identify the parameters  $\alpha$  and  $\beta$ . If we write the equation for the appendage modal coordinate from (10.3-5), we obtain

$$\ddot{\eta} + \frac{\lambda}{\mu'} \dot{\eta} + \frac{\nu}{\mu'} \eta = \left(-\frac{k_e}{\mu'}\right) \ddot{\theta}_1 \quad (10.4-1)$$

Defining

$$\begin{aligned} \sigma_F &= \sqrt{\frac{\nu}{\mu'}} \\ \beta_F &= \frac{\lambda}{2\sqrt{\nu\mu'}} \\ \delta &= -\frac{k_e}{\mu'} \end{aligned} \quad (10.4-2)$$

we can write (10.4-1) as

$$\ddot{\eta} + 2\beta_F \sigma_F \dot{\eta} + \sigma_F^2 \eta = \delta \ddot{\theta}_1 \quad (10.4-3)$$

which is a standard form. Hence, dividing (10.3-13) by  $\mu'$ , we can write the CE as

$$\sum_{j=0}^5 f_j \dot{s}^j = 0 \quad (10.4-4)$$

where

$$\begin{aligned} f_0 &= a_I \sigma_F^2 \\ f_1 &= a_P \sigma_F^2 + 2\beta_F a_I \sigma_F \\ f_2 &= a_D \sigma_F^2 + 2\beta_F a_P \sigma_F + a_I \\ f_3 &= \sigma_F^2 + 2\beta_F a_D \sigma_F + a_P \\ f_4 &= 2\beta_F \sigma_F + a_D \\ f_5 &= 1 - \frac{\delta^2}{I} ; \quad \frac{\delta^2}{I} = \frac{k_e^2}{2k_3 \mu'} \end{aligned} \quad (10.4-5)$$

ααα

Now, from the three instrument characteristics  $\sigma_F$ ,  $f_F$  and  $\frac{\delta^2}{I}$  we can identify two as the  $\alpha$ ,  $\beta$  parameters. Usually,  $f_F$  is assumed, hence,  $\sigma_F$  and  $\frac{\delta^2}{I}$  are the candidates for the  $\alpha$ ,  $\beta$  parameters. The rest of the quantities in  $f_j$  are to be chosen including  $f_F$ . The control gains  $a_p$ ,  $a_D$ ,  $a_I$  can be chosen in any appropriate manner bearing in mind that the choice of these should give a stable system if the appendage were rigid, i.e., the system

$$s^3 + a_D s^2 + a_P s + a_I = 0 \quad (10.4-6)$$

should be a stable system for any choice of the gains. Following the procedure for the parameter plane analysis we recast  $f_j$  in terms of  $\sigma_F$  and  $\frac{\delta^2}{I}$  as follows:

$$f_j = \bar{\alpha}_j \sigma_F^2 + \bar{\beta}_j \sigma_F + \bar{\gamma}_j \frac{\delta^2}{I} + \bar{\xi}_j \quad (10.4-7)$$

Table 7 shows the coefficients  $\bar{\alpha}_j$ ,  $\bar{\beta}_j$ ,  $\bar{\gamma}_j$  and  $\bar{\xi}_j$ . Then obtain the two algebraic equations

$$\begin{aligned} A_1 \sigma_F^2 + A_2 \sigma_F + A_3 \frac{\delta^2}{I} + A_4 &= 0 \\ B_1 \sigma_F^2 + B_2 \sigma_F + B_3 \frac{\delta^2}{I} + B_4 &= 0 \end{aligned} \quad (10.4-8)$$

Solution of these equations for various values of  $f$ ,  $\omega_n$  gives  $\sigma_F$  and  $\frac{\delta^2}{I}$ . Figures 14 through 17 show these parameter plane curves. Figure 14 shows the unstable region; Fig. 15 shows the  $f$ -curves for  $f = 0, 0.1, \dots, 1$ ; Fig. 16 shows a few curves of Fig. 15 for clarity, and Fig. 17 shows the effect of varying the  $f_F$ . As  $f_F$  decreases the unstable region expands.

To illustrate the use of these curves, refer to Fig. 16. If a designer wants to keep  $\sigma_F > 0.6$  and damping ratio  $\zeta$  between  $\zeta = 0.1$  and  $0.2$  he could, for instance, work in the elbow region shaded in the diagram. By adjusting the system parameters (essentially the instrument characteristics) he could obtain point A or B if he so chooses.

For the two body system considered we see the parameter plane lending itself completely to the designer's choice of the system parameters. We assumed only the control gains and  $\zeta_F$  of the instrument in the whole system in plotting these curves. All other system characteristics especially the flexible instrument can be varied over a wide range of choice.

### 10.5 Three-Body System

The characteristic equation for the three-body system is given in (10.3-9) through (10.3-11). As in the case of a two-body system we will try to identify two parameters,  $\sigma_F$  and  $\zeta_F$ . The equation for the appendage modal coordinate from (10.3-5) is

$$\ddot{\eta} + \frac{\lambda}{\mu'} \dot{\eta} + \frac{\nu}{\mu'} \eta = \left(-\frac{b_g}{\mu'}\right) \ddot{\theta}_1 + \left(\frac{\mu_{\psi\phi} - \mu_{\phi\psi}}{\mu'}\right) \ddot{\theta}_2 \quad (10.5-1)$$

Defining

$$\begin{aligned} \sigma_F &= \sqrt{\frac{\nu}{\mu'}} \\ \zeta_F &= \frac{\lambda}{2\sqrt{\nu\mu'}} \\ \delta &= \frac{\mu_{\psi\phi} - \mu_{\phi\psi}}{\mu'} \end{aligned} \quad (10.5-2)$$

the equation (10.5-1) becomes

$$\ddot{\eta} + 2\zeta_F\sigma_F\dot{\eta} + \sigma_F^2\eta = \left(-\frac{b_g}{\mu'}\right) \ddot{\theta}_1 + \delta \ddot{\theta}_2 \quad (10.5-3)$$

This equation is similar to the case of a two-body system (see (10.4-3)) except that there is an additional term on the right side due to the added motion in the system. This indicates that the appendage motion is controlled by both motions  $\theta_1$  and  $\theta_2$ .

Examination of the coefficients  $f_j$  in (10.3-10) tells us that  $f_j$  cannot be expressed in terms of only two flexibility parameters  $\alpha$ ,  $\beta$  and  $\zeta_F$  of the instrument, and the control gains. There are additional terms which would

not disappear. Thus, for the three-body system the coefficients do not simplify as in the case of a two-body system. Nevertheless, a parameter plane study can be carried out for any chosen pair of parameters. All quantities except the two parameters must then be specified. The usefulness of the curves depends on what and how many quantities are specified in the system. Ideally we would like to leave the entire flexible instrument unspecified (except  $\mathcal{J}_F$ ). In the following three different forms are shown for writing the  $f_j$  in terms of some possible parameters. In all cases the CE is the same, i.e., (10.3-9).

1) Form One

$$f_0 = \sigma_F^2 \gamma_1$$

$$f_1 = \sigma_F^2 \gamma_2 + 2 \int_F \sigma_F \gamma_1$$

$$f_2 = \sigma_F^2 \gamma_3 + 2 \int_F \sigma_F \gamma_2 + \gamma_1$$

$$f_3 = \sigma_F^2 \gamma_4 + 2 \int_F \sigma_F \gamma_3 + \gamma_2$$

$$f_4 = \sigma_F^2 \gamma_5 + 2 \int_F \sigma_F \gamma_4 + \gamma_3$$

$$f_5 = \sigma_F^2 \gamma_6 + 2 \int_F \sigma_F \gamma_5 + \gamma_4'$$

$$f_6 = \sigma_F^2 \gamma_7 + 2 \int_F \sigma_F \gamma_6 + \gamma_5'$$

$$f_7 = 2 \int_F \sigma_F \gamma_7 + \gamma_6'$$

$$f_8 = \gamma_7'$$

(10.5-4)

in which  $\sigma_F$  is one parameter and the normalized mass  $\frac{m_F}{M}$  of the instrument is the other parameter. The  $\gamma_i, \gamma_i'$  are functions (polynomials) of the normalized mass. All other physical attributes except these parameters have to be specified.

If the  $f_j$  are written in the form

$$f_j = \bar{\alpha}_j \sigma_F^2 + \bar{\beta}_j \sigma_F + \bar{\gamma}_j \delta'^2 + \bar{\xi}_j \delta' + \bar{\psi}_j \quad (10.5-5)$$

where

$$\delta' = \frac{M\phi - M\phi\gamma_2}{\sqrt{2\gamma_2} \sqrt{M'}} \quad (10.5-6)$$

then the  $\bar{\alpha}_j, \bar{\beta}_j, \bar{\gamma}_j$ , etc. are given in Tables 8 and 9.

Table 8 is for the second form and Table 9 is for the third form for these  $f_j$  of the CE.

$$a_{p1} = \frac{k_{11}}{2b_3} \quad ; \quad a_{d1} = \frac{k_{12}}{2b_3} \quad ; \quad a_{\Sigma 1} = \frac{k_{13}}{2b_3}$$

$$a_{p2} = \frac{k_{21}}{2\gamma_2} \quad ; \quad a_{d2} = \frac{k_{22}}{2\gamma_2} \quad ; \quad a_{\Sigma 2} = \frac{k_{23}}{2\gamma_2}$$

$$N = \frac{b_6}{2\sqrt{\gamma_2} b_3} \quad ; \quad \Omega' = \frac{m_{R1}}{M} \frac{b_2}{\sqrt{2b_3}} \frac{M\phi}{\sqrt{M'}}$$

$$\Omega = \Omega' + \sqrt{\frac{\gamma_2}{b_3}} \delta' \quad (10.5-7)$$

In exactly a similar manner as in the case of a two-body system the control gains are chosen to stabilize a full three rigid-body system. Since  $0 \leq N \leq 1$  the rigid system can be stabilized such that it is stable for any value of  $N$ . The value  $N$  depends on only mass distribution. In the forms two and three  $\omega$  or  $\omega'$  is an additional parameter which is dependent on the parameter  $\delta'$ . Hence, it is not very useful to plot planar curves of  $\sigma_F$  and  $\delta'$ .

In the present work form one is used to plot the parameter plane curves. All quantities specified are normalized with respect to some system quantities. Figures 18 and 19 show these curves. The unstable region is shown shaded. In Fig. 18 the effect of varying  $\zeta_F$  is also seen. Due to the existence of multiple roots there are multiple  $\zeta$ -curves. Figure 19 shows some of these  $\zeta$ -curves.

## 10.6 Summary and Conclusions

In the case of a two-body system the parameter plane proved to be a very elegant and powerful method to study the flexibility characteristics of the class of instruments that may be fitted to the shuttle of a multi-purpose spacecraft. Once the model is specified and the control gains are obtained in terms of ratios to some system quantity, the parameter plane curves can be plotted for any given value of the modal damping  $\zeta_F$ . No other data need be specified. This offers plenty of freedom in designing the flexible instrument. Also, the physical characteristics of the shuttle and the IPM can also be adjusted. Indeed the parameter plane curves are very useful to the designer if the system is represented by a two-body model.

In the case of a three-body system complications arise and separating the flexibility characteristics of the instruments into two parameters is not possible. There must be three parameters (excluding  $\zeta_F$ ) to portray the flexibility of the instruments. But two of them are dependent through the physical characteristics of the instruments. Hence, even a three dimensional plot is not very useful. One could plot sets of two parameter plane curves for a wide range of the third parameter. But this may not be convenient or reasonably straightforward. The alternate approach is to



specify as much data as is essential regarding the shuttle and the IPM and try to express the flexibility characteristics that are most variable from the designer's point of view as parameters. This will require specification of some physical attributes of the instrument itself. This undoubtedly lessens the freedom of design of the instruments. Some ingenuity is called for in choosing the proper parameters so that least number of data about the instrument need only be specified and the curves obtained are most useful in direct application in designing the rest of the characteristics of the instruments. In short, three-body system leads to limitations in the use of the parameter plane curves.

TABLE 7. COEFFICIENTS OF THE CHARACTERISTIC  
Equation for parameter Plane Analysis  
for a Two-Body System

j	COEFFICIENT			
	$\bar{\alpha}_j$	$\bar{\beta}_j$	$\bar{\gamma}_j$	$\bar{\delta}_j$
0	$a_I$	0	0	0
1	$a_P$	$2\zeta_F a_I$	0	0
2	$a_D$	$2\zeta_F a_P$	0	$a_I$
3	1	$2\zeta_F a_D$	0	$a_P$
4	0	$2\zeta_F$	0	$a_D$
5	0	0	-1	1

Table 8 . Coefficients of the CE for  
Parameter Plane Analysis for  
a Three-Body System ; Form 2.

j	Coefficients				
	$\overline{\alpha}_j$	$\overline{\beta}_j$	$\overline{\gamma}_j$	$\overline{\delta}_j$	$\overline{\psi}_j$
0	$a_{I1} a_{I2}$	0	0	0	0
1	$a_{P1} a_{I2} +$ $a_{I1} a_{P2}$	$2 f_F \overline{\alpha}_0$	0	0	0
2	$a_{P1} a_{P2} +$ $a_{D1} a_{I2} +$ $a_{I1} a_{D2}$	$2 f_F \overline{\alpha}_1$	0	0	$\overline{\alpha}_0$
3	$a_{I1} + a_{I2} +$ $a_{P1} a_{D2} +$ $a_{P1} a_{P2}$	$2 f_F \overline{\alpha}_2$	0	0	$\overline{\alpha}_1$
4	$a_{P1} + a_{P2} +$ $a_{D1} a_{D2}$	$2 f_F \overline{\alpha}_3$	0	0	$\overline{\alpha}_2$
5	$a_{D1} + a_{D2}$	$2 f_F \overline{\alpha}_4$	$-a_{I1}$	0	$\overline{\alpha}_3 - a_{I2} n^2$
6	$1 - n^2$	$2 f_F \overline{\alpha}_5$	$-a_{P1}$	0	$\overline{\alpha}_4 - a_{P2} n^2$
7	0	$2 f_F \overline{\alpha}_6$	$-a_{D1}$	0	$\overline{\alpha}_5 - a_{D2} n^2$
8	0	0	$-1$	$2 N \Omega$	$\overline{\alpha}_6 - n^2$

Table 9. Coefficients of the CE for  
Parameter Plane Analysis for  
a Three-Body System, Form 3.

j	Coefficients				
	$\bar{\alpha}_j$	$\bar{\beta}_j$	$\bar{\gamma}_j$	$\bar{\delta}_j$	$\bar{\epsilon}_j$
0	$a_{I1} a_{I2}$	0	0	0	0
1	$a_{P1} a_{I2} + a_{I1} a_{P2}$	$2\beta_F \bar{\alpha}_0$	0	0	0
2	$a_{P1} a_{P2} + a_{D1} a_{I2} + a_{I1} a_{D2}$	$2\beta_F \bar{\alpha}_1$	0	0	$\bar{\alpha}_0$
3	$a_{I1} + a_{I2} + a_{P1} a_{D2} + a_{D1} a_{P2}$	$2\beta_F \bar{\alpha}_2$	0	0	$\bar{\alpha}_1$
4	$a_{P1} + a_{P2} + a_{D1} a_{D2}$	$2\beta_F \bar{\alpha}_3$	0	0	$\bar{\alpha}_2$
5	$a_{D1} + a_{D2}$	$2\beta_F \bar{\alpha}_4$	$-(a_{I1} + \frac{f_2}{b_3} a_{I2})$	$-2\alpha' \sqrt{\frac{f_2}{b_3}} a_{I2}$	$\bar{\alpha}_3 - \alpha'^2 \bar{\alpha}_{I2}$
6	$1 - N^2$	$2\beta_F \bar{\alpha}_5$	$-(a_{P1} + \frac{f_2}{b_3} a_{P2})$	$-2\alpha' \sqrt{\frac{f_2}{b_3}} a_{P2}$	$\bar{\alpha}_4 - \alpha'^2 \bar{\alpha}_{P2}$
7	0	$2\beta_F \bar{\alpha}_6$	$-(a_{D1} + \frac{f_2}{b_3} a_{D2})$	$-2\alpha' \sqrt{\frac{f_2}{b_3}} a_{D2}$	$\bar{\alpha}_5 - \alpha'^2 \bar{\alpha}_{D2}$
8	0	0	$-(1 + \frac{f_2}{b_3} \sqrt{f_2/b_3})$	$-2\alpha' (\sqrt{\frac{f_2}{b_3}} - N)$	$\bar{\alpha}_6 - \alpha'^2$

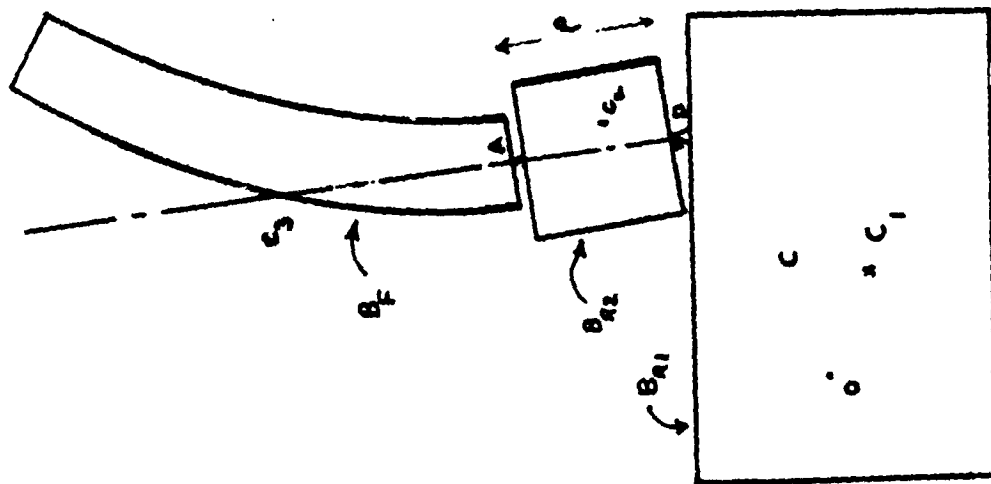


Fig. 12. A Typical Model of a Multi-Purpose Spacecraft

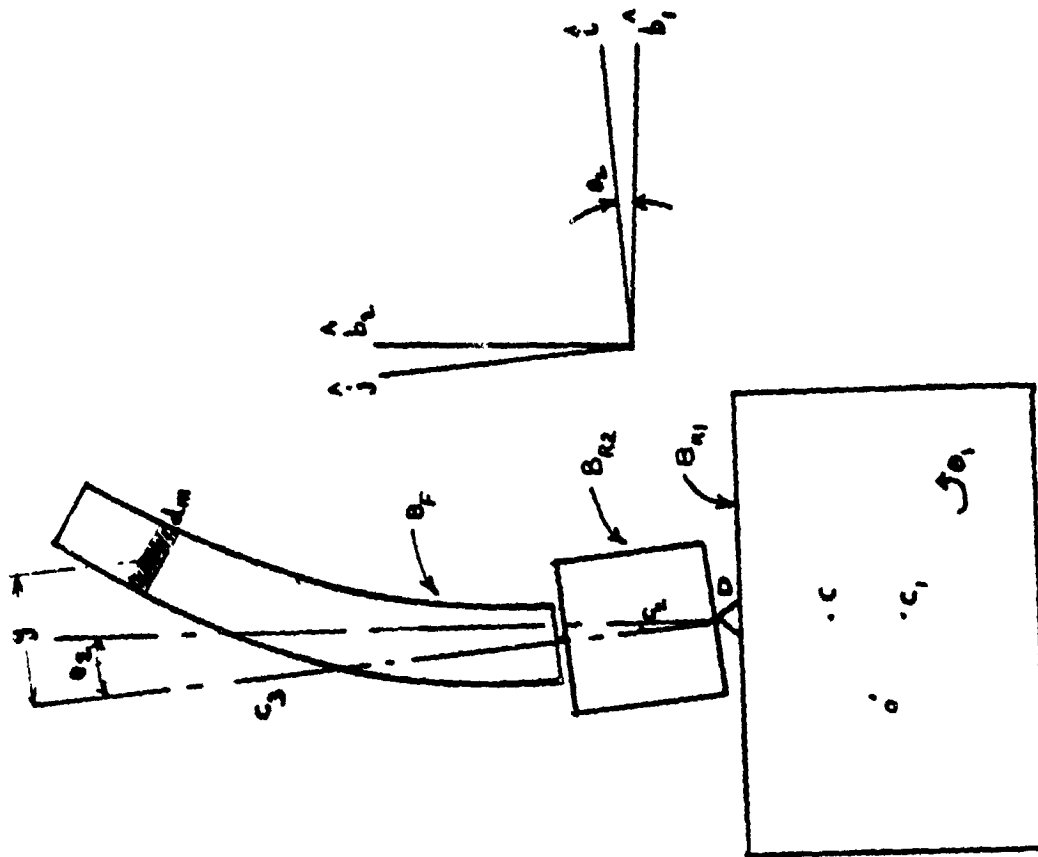


Fig. 13. A Symmetric Model of a Multi-Purpose Spacecraft

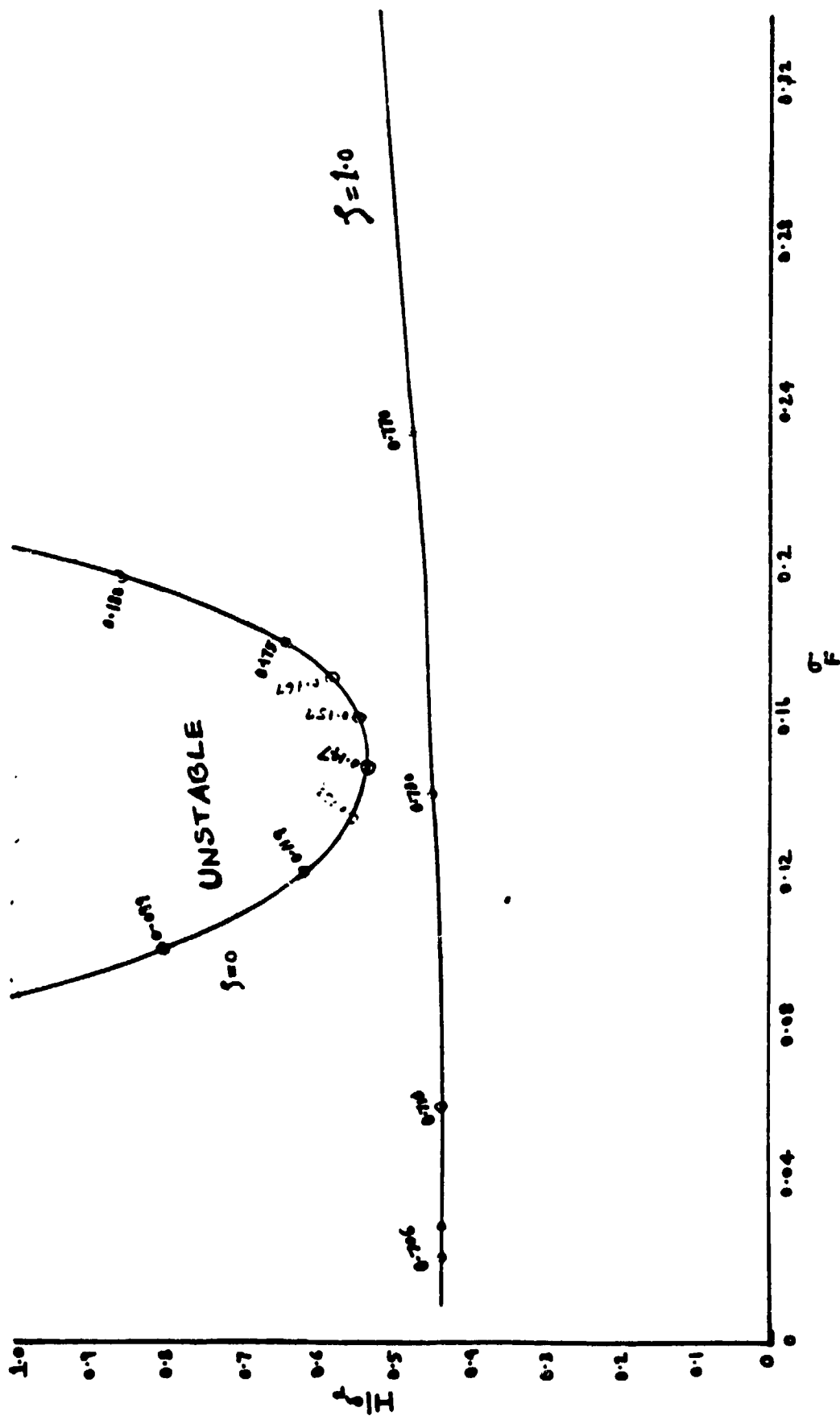


Fig. 14. Stability Region in the Parameter Plane for a Two-Body System for  $\zeta_F = 0.005$

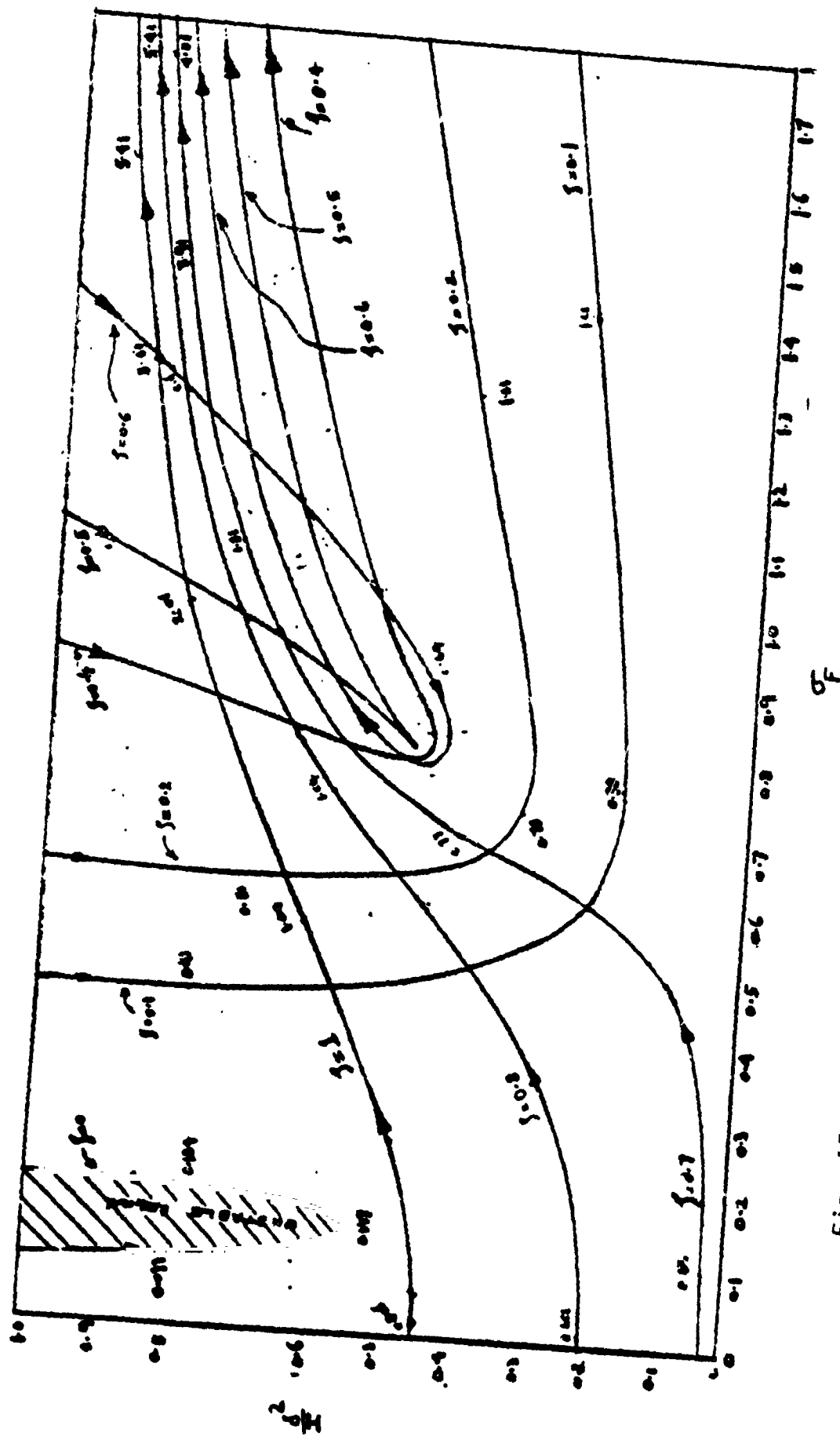


Fig. 15. Parameter Plane Curves for a Two-Body system for  $\zeta_F = 0.005$

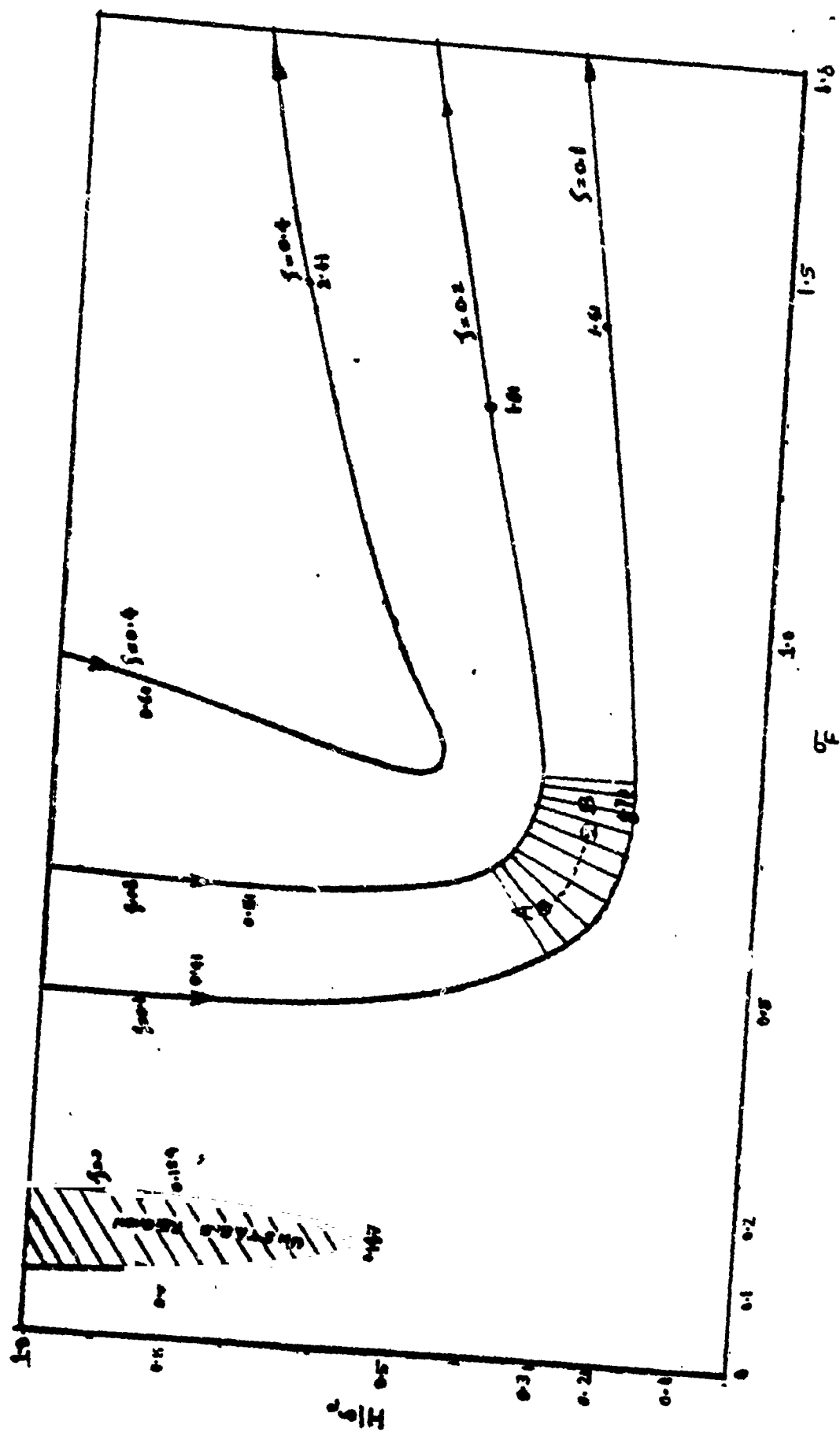


Fig. 16. Illustrating the Use of Parameter Plane Curves (taken from Fig. 15)



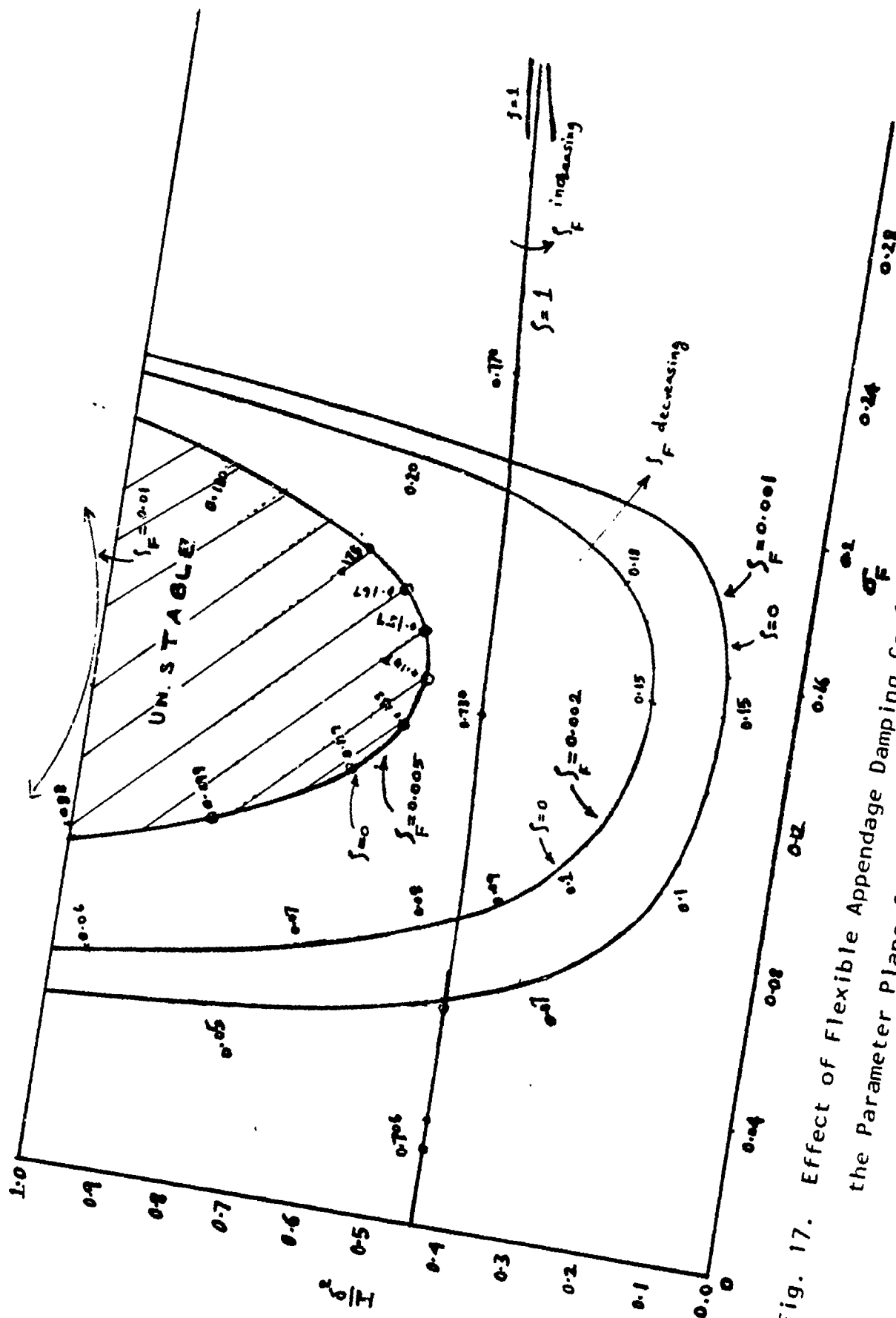


Fig. 17. Effect of Flexible Appendage Damping Coefficient  $\zeta_F$  on Stability Region in the Parameter Plane for a Two-Body System

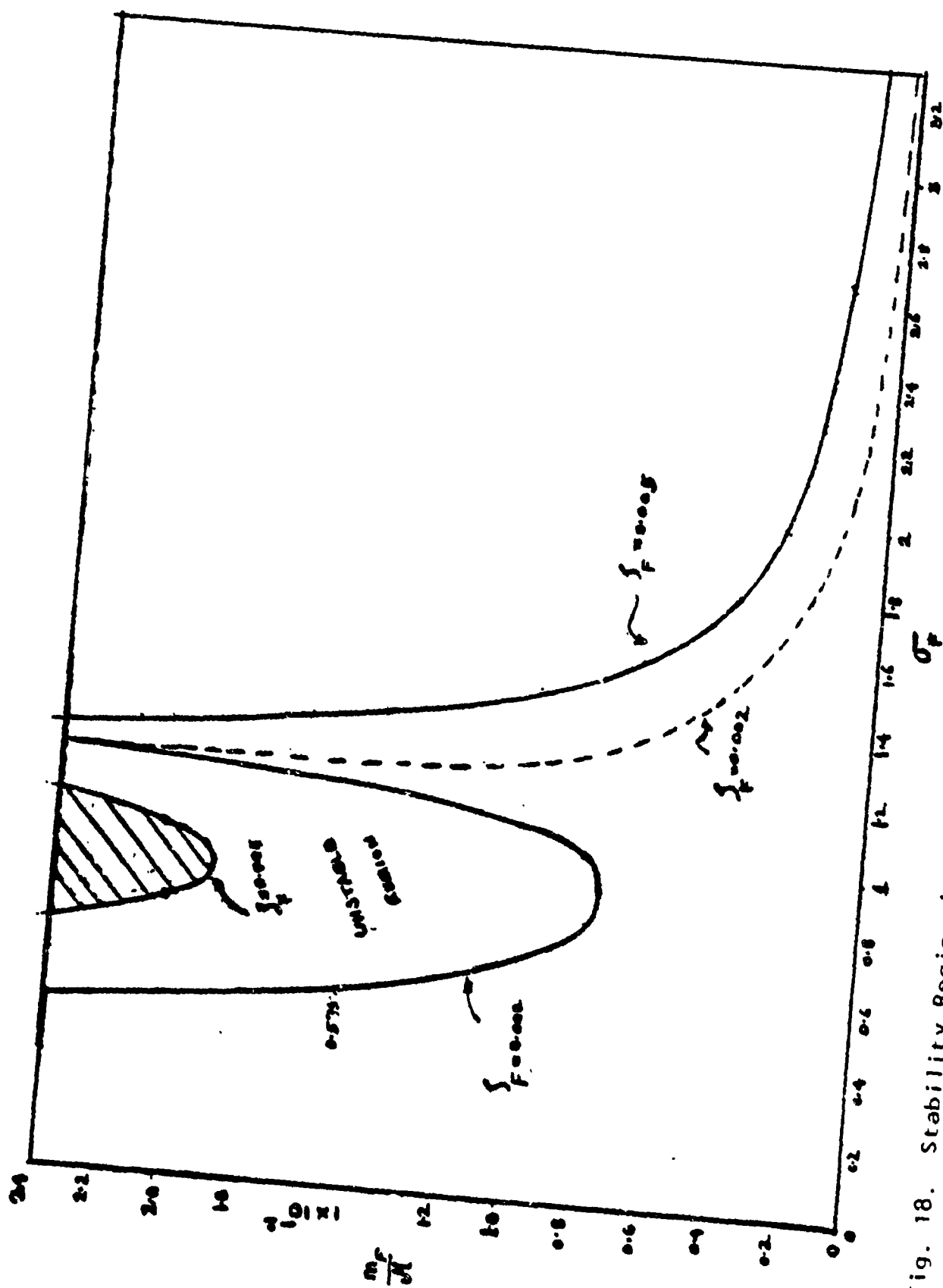


Fig. 18. Stability Region in the Parameter Plane for a Three-Body System for  $\sigma_F = 0.005$

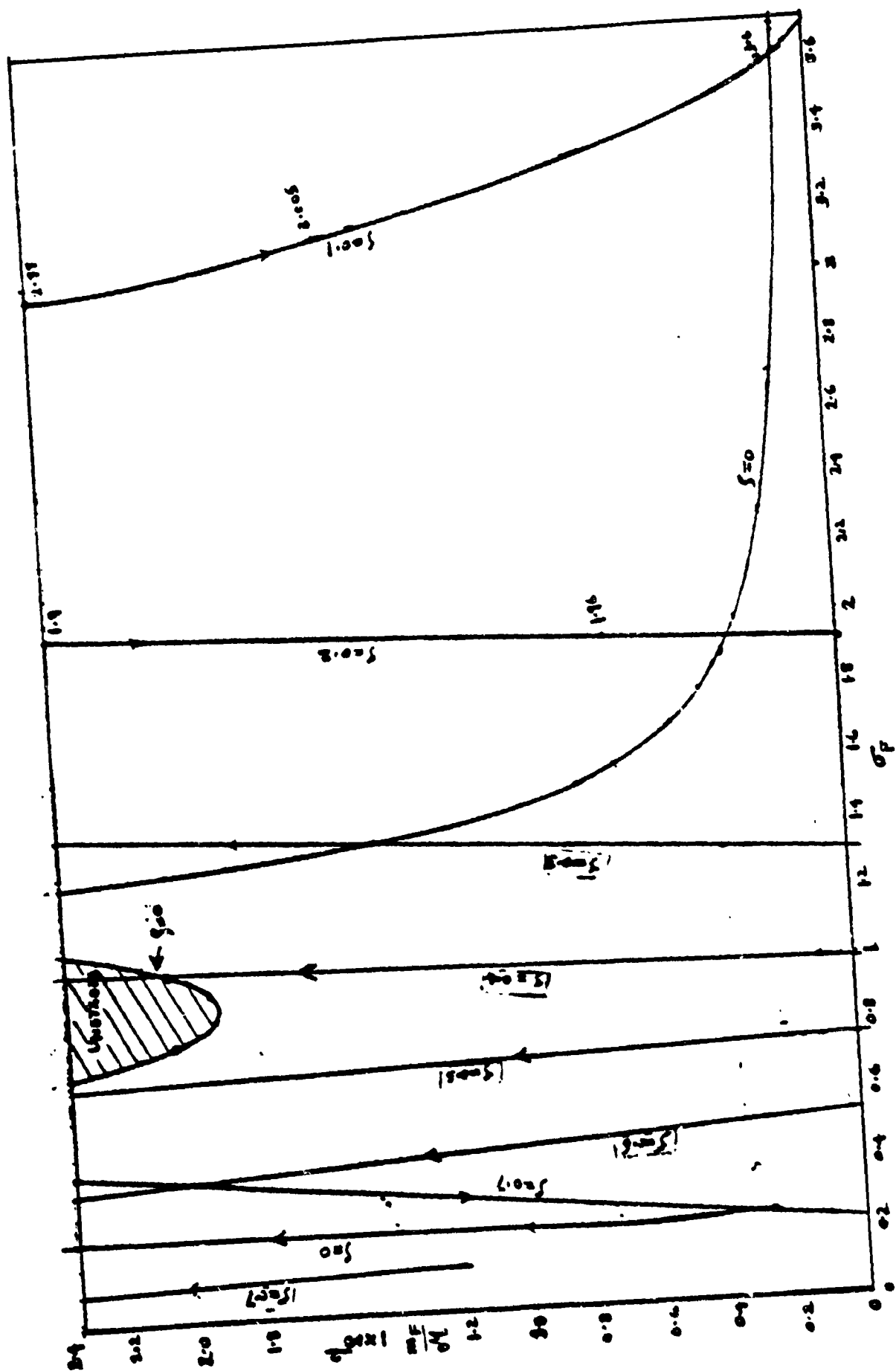


Fig. 19. Parameter Plane Curves for a Three-Body System for  $\int_F = 0.005$

8. I.A. Vishnegradsky, "Sur la Theorie generale des regulateurs", Compt., Rend. Acad. Sci., 83, 318-321 (1876).
9. D.D. Siljak, "Nonlinear Systems," Wiley, N.Y., 1969.
10. S.M. Seltzer, "Passive Stability of a Spinning Skylab", Journal of Spacecraft and Rockets, Vol. 9, 1972, pp. 651-55.
11. S.M. Seltzer, G. Schweitzer and B. Asner Jr., "Attitude Control of a Spinning Skylab," Journal of Spacecraft and Rockets, Vol. 10, 1973, pp. 200-207.
12. S.M. Seltzer and H.L. Shelton, "Specification of Spacecraft Flexibility Appendage Rigidity", Journal of Guidance and Control, Vol. 1, 1978, pp.427-432.
13. P.W. Likins, "Dynamics and Control of Flexible Space Vehicles", Jet Propulsion Laboratory, Pasadena, Calif., Rept. TR 32-1329, Rev.1, Jan. 1970.